NYU

Linear Algebra

Lecture 12

Orthogonality and Projections

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Strang Sections 4.1 – Orthogonality of the Four Subspaces and Section 4.2 - Projections

Course notes adapted from N. Hammoud's NYU lecture notes.

Projection onto a Line

Orthogonal Projection of a Vector onto a Line

Let $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$ be the direction of a line that goes through the origin, and let $\vec{b} = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$. We are interested in finding the projection $\vec{p}$ of $\vec{b}$ onto the line in the direction of $\vec{a}$.

Orthogonal Projection of a Vector onto a Line

- If $\vec{b} = \vec{a} \implies \hat{x} = 1$, and the projection of $\vec{a}$ onto $\vec{a}$ is itself $\implies P\vec{a} = \vec{a}$.
- If $\vec{b}$ is orthogonal to $\vec{a}$, then $\vec{a}^T\,\vec{b} = 0$, and therefore $\vec{p} = 0$.

To find the projection matrix P, we ask:

"What matrix should multiply $\vec{b}$ such that the outcome is $\vec{p}$?"

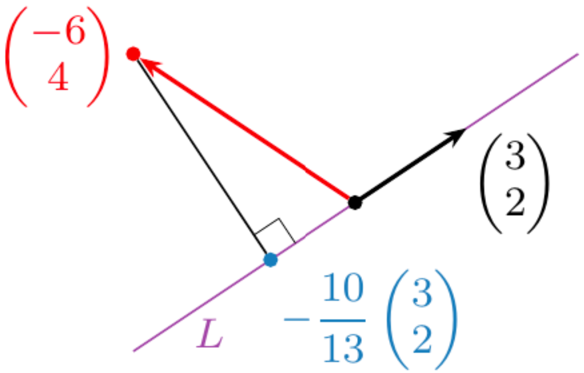
We want to find P, such that $P\vec{b} = \vec{p}$.

$$\implies P\vec{b} = \frac{\vec{a}^T\,\vec{b}}{\vec{a}^T\,\vec{a}}\,\vec{a} = \vec{a}\,\frac{\vec{a}^T\,\vec{b}}{\vec{a}^T\,\vec{a}} = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\,\vec{a}}\,\vec{b} \implies P = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\,\vec{a}}$$

Example

Compute the orthogonal projection of $x = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$ onto the line L spanned by $u = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$$y = \mathrm{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u = \frac{-18+8}{9+4} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = -\frac{10}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$



Example

Find the projection $\vec{p}$ of $\vec{b} = (1, 1, 2)$ onto $\vec{a} = (-2, 3, 1)$.

Find the projection matrix P, such that $\vec{p} = P\vec{b}$.

Projection onto a Subspace

Orthogonal Projection of a Vector onto a Subspace

Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \in \mathbb{R}^m$ be a set of linearly independent vectors, and let S be a subspace, such that:

$$S = \text{span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\} \subset \mathbb{R}^m$$

We want to project $\vec{b} \in \mathbb{R}^m$ onto S, i.e., we want to find a projection $\vec{p}$ of $\vec{b}$ onto S, such that the combination

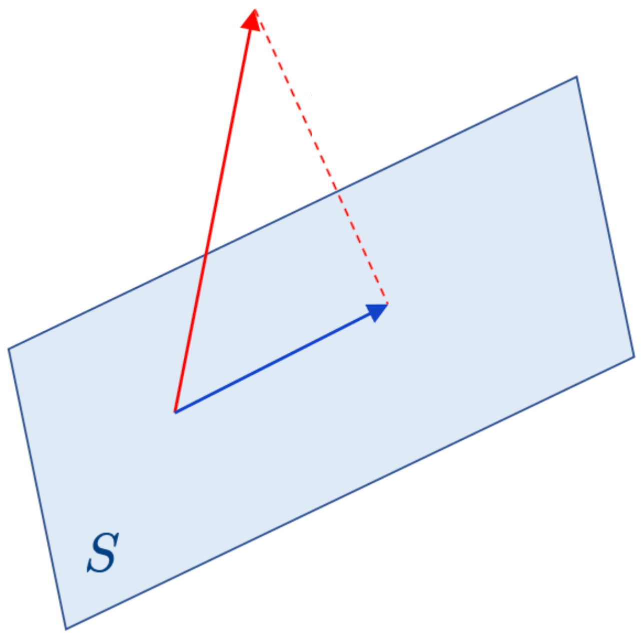
$$\vec{p} = \hat{x}_1\vec{a}_1 + \hat{x}_2\vec{a}_2 + \cdots + \hat{x}_n\vec{a}_n$$

is closest to the vector $\vec{b}$. Note that $\vec{p}$ can be written as

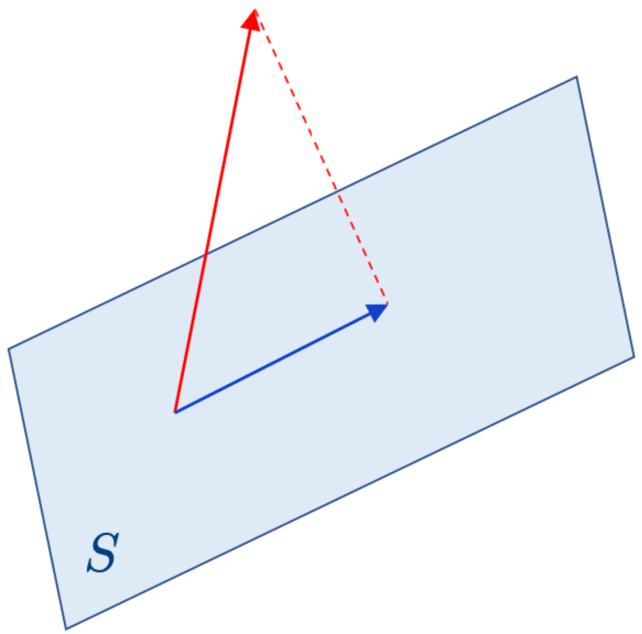
$$\vec{p} = \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n\end{bmatrix}\begin{bmatrix}\hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n\end{bmatrix} = A\vec{\hat{x}}$$

Orthogonal Projection of a Vector onto a Subspace

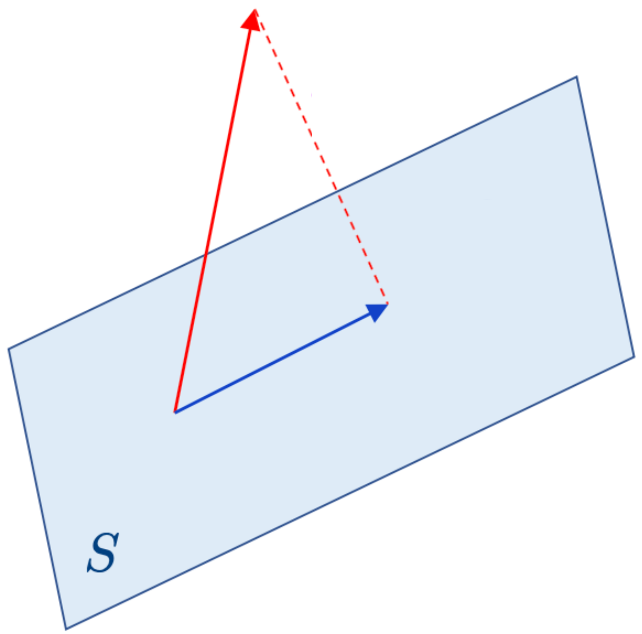
First, we must find the vector $\vec{\hat{x}}$, then we find $\vec{p}$, and then we find the projection matrix P.



Orthogonal Projection of a Vector onto a Subspace



Orthogonal Projection of a Vector onto a Subspace



When is $A^T A$ invertible?

Theorem: $A^T A$ is invertible if and only if A has linearly independent columns.

Examples

Example: Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$. Find $\vec{p}$, the projection of $\vec{b}$ onto $\operatorname{Col} A$, and find the projection matrix P.