$A \in \mathbb{R}^{47\times 32}$

Which of the following is not possible

- [x] $\text{rank}(A) = 40$ because $\text{rank} \le m$, $\text{rank} \le n$.
- [x] $\text{rank}(A) = 32$, linear system $Ax = b, b \in \mathbb{R}^{47}$ have infinite solutions ← $\text{rank} = \#\text{Var} \Rightarrow \dim(\text{Nul}(A)) = 0$
- [ ] $\text{rank}(A) = 32$, linear system $Ax = b, b \in \mathbb{R}^{47}$ have no solutions
- [x] $\text{rank}(A) = 32$, linear system $A^\top x = b, b \in \mathbb{R}^{32}$ have no solution

$Ax = b$. $A \in \mathbb{R}^{47\times 32}$, #32 Variables, #47 Equation.

$\Rightarrow \text{Nul}(A) = \{\vec{0}\} \Rightarrow$ at most 1 Solution

$A^T x = b$. $A^T \in \mathbb{R}^{32\times 47}$ # 32 Equation # 47 Variables.

$\text{rank}(A) = 32 = \#\text{Eq} \Rightarrow A^T$ is a full row rank matrix

$\Rightarrow \text{Col}(A^T) = \mathbb{R}^{32} \Rightarrow A^T x = b$ at least have 1 Solution

$A^T x = b$. $b \in \mathbb{R}^{32}$. $\text{rank}(A) = 32$ must have infinite solutions!

$\dim(\text{Nul}(A^T)) = 47 - \underset{\text{rank}}{32} = 15 > 0 \Rightarrow A^T x = b$ must have have infinit solution

Suppose that A is a 3×2 matrix whose Nullspace contains exactly one element. → $\text{Nul}(A) = \{\vec{0}\}$ → $\text{rank}(A) = 2$

Which of the following would be a basis for Row A? → $\text{Row}(A) \subseteq \mathbb{R}^2$, $\dim(\text{Row}(A)) = \text{rank}(A) = 2$ $\Rightarrow \text{Row}(A) = \mathbb{R}^2$

- ( ) $\{\vec{0}\}$
- (•) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^2$.
- ( ) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$
- ( ) Contains 1 vector but there is insufficient information to determine a basis
- ( ) Contains 2 vectors but there is insufficient information to determine a basis

Checking Subspace

- If V is a subspace. check $\vec{v}_1, \vec{v}_2 \in V \Rightarrow c_1\vec{v}_1 + c_2\vec{v}_2 \in V$
- If V is not a subspace, give me an example . shows that $\vec{v}_1, \vec{v}_2 \in V$ but $c_1\vec{v}_1 + c_2\vec{v}_2 \notin V$

Ex. $V = \{(x,y) \mid x^2 - y^2 = 0\}$ is not a subspace

$\vec{v}_1 = (1,-1)$, $\vec{v}_2 = (1,+1)$, then $\vec{v}_1 + \vec{v}_2 = (2,0) \notin V$

$2^2 - 0^2 \neq 0$

so V is not a subspace.

Ex. a) all symmetric Matrix is a subspace.

b) all polynomials like $\{f \mid f(x) = ax^2 + bx + c\}$ is a subspace

$a, b, c \in \mathbb{R}$

b) $f_1 = a_1x^2 + b_1x + c_1$, $f_2 = a_2x^2 + b_2x + c_2$

$\Rightarrow d_1f_1 + d_2f_2 = (d_1a_1 + d_2a_2)x^2 + (d_1b_1 + d_2b_2)x + (d_1c_1 + d_2c_2)$ $\} \Rightarrow V$

What is the basis of b),

basis is not unique! You just need to give an example

△ just a way to provide a basis.

- $a=1$ $b=0$ $c=0$ $f(x) = x^2$
- $a=0$ $b=1$ $c=0$ $f(x) = x$
- $a=0$ $b=0$ $c=1$ $f(x) = 1$

b) is a subspace with dim 3 and basis is $\{x^2, x, 1\}$

---

a) check symmetric matrix is subspace

What is the basis? $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightarrow a_{12} = a_{21}$

a_{11}, a_{12}, a_{22} are free. a_{21} is fixed

- $a_{11} = 1$ $a_{12} = 0$ $a_{22} = 0$ $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
- $a_{11} = 0$ $a_{12} = 1$ $a_{22} = 0$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $a_{11} = 0$ $a_{12} = 0$ $a_{22} = 1$ $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

2x2 Sym Matrix is a vector space

dim = 3

$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

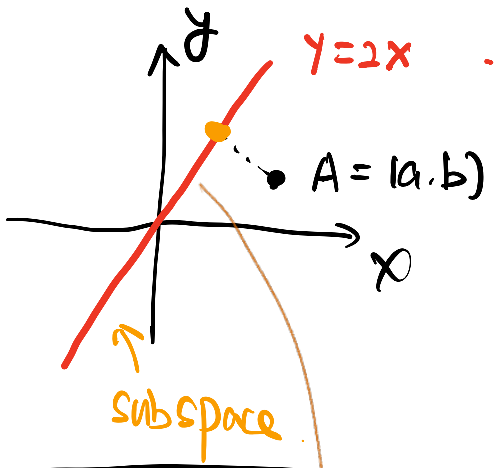
Linear Algebra

Lecture 12

Orthogonality and Projections

Dr. Ralph Chikhany

Motivation of projection



Question:

what is the nearest point on line $y=2x$ to (a,b)?

Cases that line system don't have solution.

$$\begin{cases} x = 1 \\ 2x = 1 \end{cases}$$

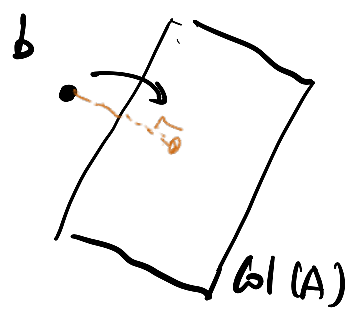
use project to guess the best solution

is all the point of $(x, 2x)$ on red line

$A=(1,1)$

$Ax = b$ That don't have a solution

$b \notin Col(A)$



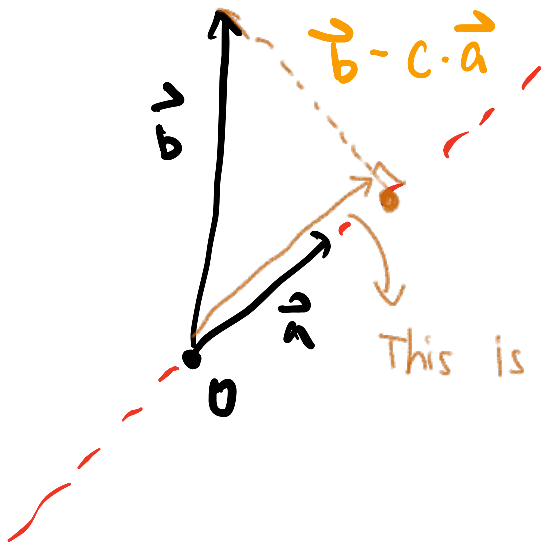
Strang Sections 4.1 – Orthogonality of the Four Subspaces and Section 4.2 - Projections

Course notes adapted from N. Hammoud's NYU lecture notes.

Projection onto a Line

Orthogonal Projection of a Vector onto a Line

Let $\vec{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$ be the direction of a line that goes through the origin, and let $\vec{b} = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$. We are interested in finding the projection $\vec{p}$ of $\vec{b}$ onto the line in the direction of $\vec{a}$.



This is $c \cdot \vec{a}$

$c \in \mathbb{R}$ I want to solve.

If $c \cdot \vec{a}$ is the projection of $\vec{b}$ onto the line. $(\vec{b} - c\cdot\vec{a}) \perp \vec{a}$

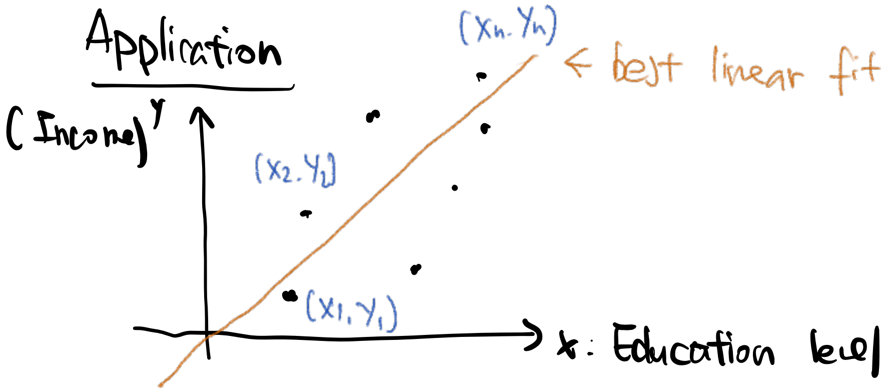
$\Rightarrow (\vec{b} - c\cdot\vec{a}) \cdot \vec{a} = 0$ by defn of orth.

$\Rightarrow \vec{b} \cdot \vec{a} - c(\vec{a} \cdot \vec{a}) = 0$

$\Rightarrow c = \dfrac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}$

The projection is $\dfrac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \vec{a}$

Application



← best linear fit.

Income = C · Education Level

$C \in \mathbb{R}$

First Vector : data of Edu level $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \vec{x}$

Second Vector: data of Income $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \vec{y}$

Question of best linear fit. $\vec{y} \simeq c \cdot \vec{x} \Rightarrow c = \dfrac{\vec{y} \cdot \vec{x}}{\vec{x} \cdot \vec{x}}$

$\dfrac{\vec{y} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \vec{x}$

Orthogonal Projection of a Vector onto a Line

- If $\vec{b} = \vec{a} \implies \hat{x} = 1$, and the projection of $\vec{a}$ onto $\vec{a}$ is itself $\implies P\vec{a} = \vec{a}$.
- If $\vec{b}$ is orthogonal to $\vec{a}$, then $\vec{a}^T\,\vec{b} = 0$, and therefore $\vec{p} = 0$.

To find the projection matrix P, we ask:

"What matrix should multiply $\vec{b}$ such that the outcome is $\vec{p}$?"

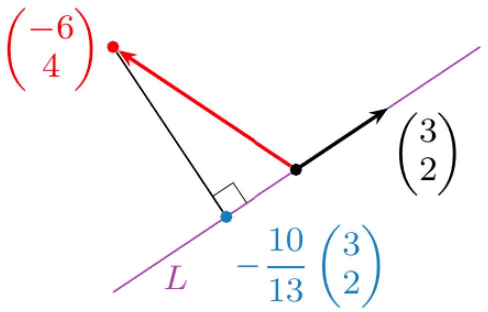
We want to find P, such that $P\vec{b} = \vec{p}$.

$$\implies P\vec{b} = \frac{\vec{a}^T\,\vec{b}}{\vec{a}^T\,\vec{a}}\,\vec{a} = \vec{a}\,\frac{\vec{a}^T\,\vec{b}}{\vec{a}^T\,\vec{a}} = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\,\vec{a}}\,\vec{b} \implies P = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\,\vec{a}}$$

Example

Compute the orthogonal projection of $x = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$ onto the line L spanned by $u = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$$y = \mathrm{proj}_L(x) = \frac{x \cdot u}{u \cdot u} u = \frac{-18+8}{9+4} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = -\frac{10}{13} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$



Example

Find the projection $\vec{p}$ of $\vec{b} = (1, 1, 2)$ onto $\vec{a} = (-2, 3, 1)$.

Find the projection matrix P, such that $\vec{p} = P\vec{b}$.

Projection onto a Subspace

Orthogonal Projection of a Vector onto a Subspace

Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \in \mathbb{R}^m$ be a set of linearly independent vectors, and let S be a subspace, such that:

$\vec{a}_1, \cdots \vec{a}_n$ is the basis of S

$$S = \text{span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\} \subset \mathbb{R}^m$$

$A \in \mathbb{R}^{m \times n}$: $n \leq m$

We want to project $\vec{b} \in \mathbb{R}^m$ onto S, i.e., we want to find a projection $\vec{p}$ of $\vec{b}$ onto S, such that the combination

$$\vec{p} = \hat{x}_1 \vec{a}_1 + \hat{x}_2 \vec{a}_2 + \cdots + \hat{x}_n \vec{a}_n$$

$\vec{p}$ is nearest to my $\vec{b}$

is closest to the vector $\vec{b}$. Note that $\vec{p}$ can be written as

$$\vec{p} = [\vec{a}_1 \ \vec{a}_2 \ \ldots \ \vec{a}_n] \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{bmatrix} = A\vec{\hat{x}}$$

$\Leftarrow S = Col(A)$

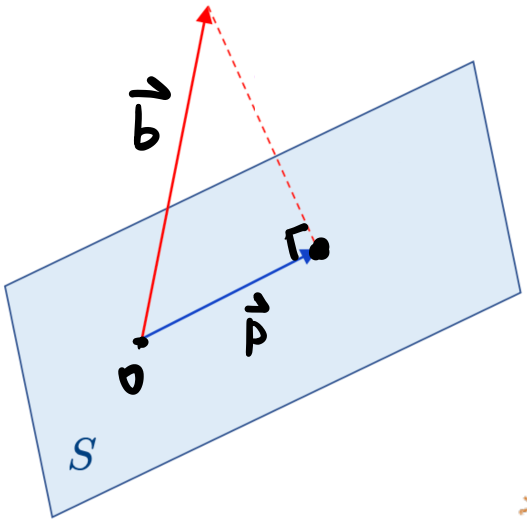
<u>Remark</u>: if $b \notin Col(A)$, $Ax = b$ don't have a solution; find p (nearest to b) $\in Col(A)$ so that $Ax = b$ will have a solution

Orthogonal Projection of a Vector onto a Subspace

"$\text{Nul}(A^T) = (\text{Col}(A))^\perp$"

$\Downarrow$

First, we must find the vector $\vec{\hat{x}}$, then we find $\vec{p}$, and then we find the projection matrix P.



- $\vec{b} - \vec{p} \perp S$ $\Rightarrow$ $\vec{b} - \vec{p} \perp \text{Col}(A)$ $(\vec{b} - \vec{p} \in \text{Nul}(A^T))$ "left Null space"
- $\vec{p} \in S$ $\Rightarrow$ $\vec{p} \in \text{Col}(A)$

$\vec{p} \in \text{Col}(A)$, $\Rightarrow$ $\vec{p} = A\vec{x}$ find the "x"

$\Rightarrow (\vec{b} - A\vec{x}) \perp S$ $(\vec{b} - A\vec{x} \in \text{Nul}(A^T))$

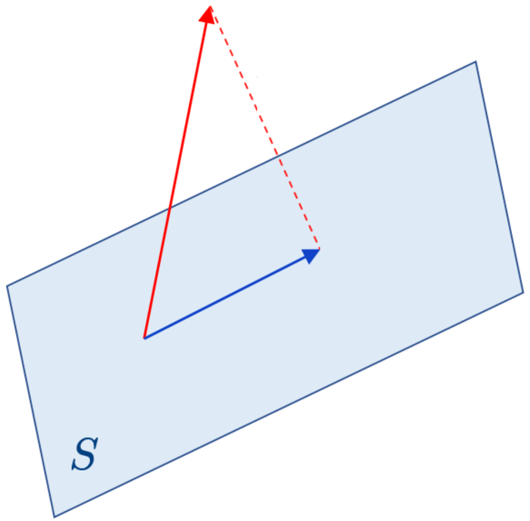
$\Rightarrow A^T \cdot (\vec{b} - A\vec{x}) = 0$ (by defn of left Null space)

$\Rightarrow A^T\vec{b} = \underbrace{A^T A}_{\text{square}}\vec{x}$ (*)

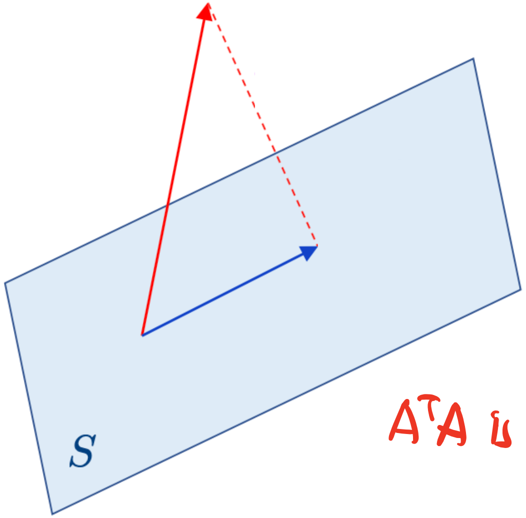
what is the size of $A \in \mathbb{R}^{m \times n}$ $(n \leq m)$

$A^T A$ is of size $\mathbb{R}^{n \times n}$ (Square matrix, Symmetric Matrix), $A^T b \in \mathbb{R}^n$

Orthogonal Projection of a Vector onto a Subspace



Orthogonal Projection of a Vector onto a Subspace



Symmetric

Square matrix, 2x2

Example. $n \leq m$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} \quad A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

A^TA is not invertible

rank$(A) = 1$ rank$(A^TA) = 1$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \quad A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 9 \end{bmatrix}$$

A^TA is invertible

rank$(A^TA) = 2$.

rank$(A) = 2$

rank(A) = rank(A^TA)

When is A^TA invertible? Nout Required

$n \times n$

Theorem: A^TA is invertible if and only if A has linearly independent columns.

$\text{rank}(A^TA) = n$ $\qquad$ $\text{rank}(A) = n$. A is full Column

I'll prove $\text{Rank}(A) = \text{rank}(A^TA)$

Firstly I'll show $\text{Nul}(A) = \text{Nul}(A^TA)$ $\Rightarrow \dim(\text{Nul}(A)) = \dim(\text{Nul}(A^TA))$

This means for a vector $x \in \mathbb{R}^n$

$$\Downarrow$$

$$Ax = 0 \iff A^TAx = 0 \qquad \text{rank}(A) = \text{rank}(A^TA)$$

① $Ax = 0 \Rightarrow A^TAx = 0$

$$A^TAx = A^T(Ax) = A^T\vec{0} = \vec{0}$$

② $A^TAx = 0 \Rightarrow Ax = 0$

$$A^TAx = 0 \Rightarrow x^T(A^TAx) = x^T0 = 0 \Rightarrow \underbrace{(Ax)^T Ax}_{\|Ax\|^2} = 0 \Rightarrow Ax = \vec{0}$$

$$x^T(A^TAx) = (Ax)^T Ax$$

$$(Ax)^T = x^TA^T$$

$\|Ax\|^2$: length of Ax is 0

Examples

Example: Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$. Find $\vec{p}$, the projection of $\vec{b}$ onto Col A, and find the projection matrix P.

If A is full column rank, Then $A^T A$ is invertible

finding projection $\left.\begin{array}{l} \vec{b} - \vec{p} \perp \text{Col}(A) \\ \vec{p} \in \text{Col}(A) \end{array}\right\} \Rightarrow$ Solving $A^T \vec{b} = A^T A \vec{x}$

$\vec{p} = A\vec{x}$

$\Rightarrow \vec{x} = (A^T A)^{-1} A^T \vec{b}$

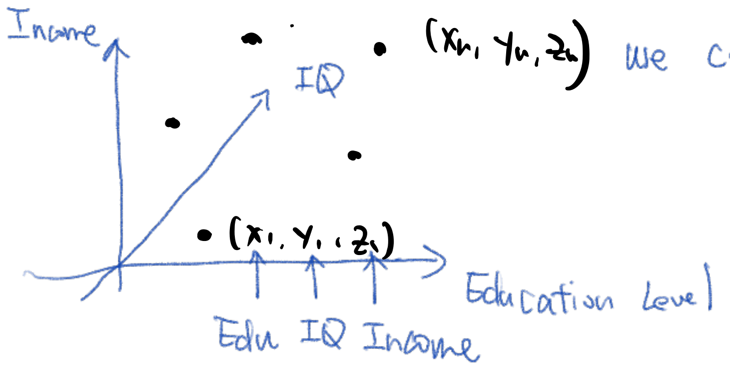
$\Rightarrow \vec{p} = A\vec{x}$

$= \underbrace{A (A^T A)^{-1} A^T}_{\text{projection matrix}} \vec{b}$

Solution $\vec{p} = P\vec{b}$. $P = A(A^T A)^{-1} A^T$

$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$

$P = A \cdot \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}^{-1} A^T = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$



we c_1 IQ + c_2 Education Level $\approx$ Income.

Find the best (c_1, c_2)

Vector 1, $\vec{a_1}$: Education level $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Vector 2 $\vec{a_2}$: IQ level $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

Vector 3, $\vec{b}$: Income. $\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$

$c_1 \cdot \vec{a_1} + c_2 \cdot \vec{a_2}$ best guess of $\vec{b}$

$A = [\vec{a_1}, \vec{a_2}] \in \mathbb{R}^{n \times 2}$ ← how many features I use.

n: number of data

$\vec{a_1}$: all the Edu $\vec{a_2}$: all the IQ

Projection Matrix, $A \underbrace{(A^T A)^{-1} A^T \vec{b}}_{\text{this is the coefficient}}$ ← $c_1 \vec{a_1} + c_2 \vec{a_2}$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = (A^T A)^{-1} A^T \vec{b}$$

$\mathbb{R}^{2\times 2}$ $\mathbb{R}^{2\times n}$ $\mathbb{R}^n \rightarrow \mathbb{R}^2$

Review for Today.

$Ax = b$ may not have solution # Var < # Eq

$n < m$

$b \notin Col(A)$. Find $\vec{p} \in Col(A)$

$\vec{p}$ is the nearest one to $\vec{b}$ $\Leftarrow$ $\vec{p} - \vec{b} \perp Col(A)$

<u>Answer</u> : Projection Matrix . $\vec{P} = A(A^TA)^{-1}A^T b$

(by $Ax = b$, left multiply A^T

$A^TAx = A^Tb$ A^TA is a square

$\Rightarrow x = (A^TA)^{-1}A^Tb$)

$rank(A^TA) = rank(A)$

So if A is full Column rank / column vec are linear indep.

then A^TA is invertible.

<u>Applications</u> : (linear regression / Econometrics)