Linear Algebra

Lecture 11

The Four Fundamental Subspaces

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Strang Sections 3.5 – Dimensions of the Four Subspaces

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed)
and N. Hammoud's NYU lecture notes.

Matrix Subspaces

The Subspaces Associated with a Matrix

Consider an $m \times n$ matrix $A = [\vec{a}_1 \quad \vec{a}_2 \quad \ldots \quad \vec{a}_n]$, where $\vec{a}_i \in \mathbb{R}^m$ $(1 \leq i \leq n)$.
Then, the four fundamental subspaces associated with A are:

(1) The column space Col A=span{pivot columns}$\subset \mathbb{R}^m$

(2) The row space $\text{Row}\, A = \text{Col}\, A^T \subset \mathbb{R}^n$

(3) The nullspace $\text{Nul}\, A = \left\{ \vec{x} \middle| A\vec{x} = \vec{0} \right\} \subset \mathbb{R}^n$

(4) The left nullspace $\text{Nul}\, A^T = \left\{ \vec{y} \middle| A^T\vec{y} = \vec{0} \right\} \subset \mathbb{R}^m$

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If A is and $m \times n$ matrix with rank r, then

(1) $\dim(\text{Col}\, A) = r$

(2) $\dim(\text{Row}\, A) = r$

(3) $\dim(\text{Nul}\, A) = n - r$

(4) $\dim(\text{Nul}\, A^T) = m - r$

Example

Find a basis and dimension of the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix}.$$

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Orthogonality

Recall

The dot product of two vectors $\vec{v} = (v_1, v_2, \ldots, v_n)$ and $\vec{w} = (w_1, w_2, \ldots, w_n)$ in $\mathbb{R}^n$ is given by

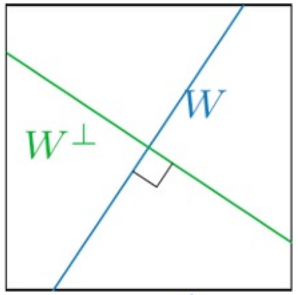
$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{i=1}^{n} v_i w_i.$$

Theorem: $\vec{v} \cdot \vec{w} = 0$ if and only if $\vec{v}$ and $\vec{w}$ are orthogonal.

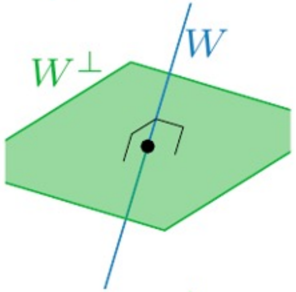
Orthogonal Subspaces

Two subspaces V and W of a vector space are orthogonal if every vector $\vec{v} \in V$ is orthogonal to every vector $\vec{w} \in W$. That is,

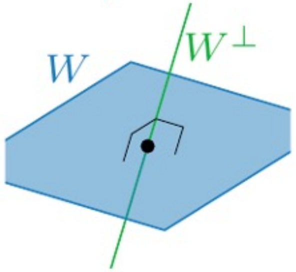
$$\vec{v}^T \vec{w} = 0 \quad \text{for all } \vec{v} \in V \text{ and all } \vec{w} \in W.$$



Every vector $\vec{x} \in \text{Nul}\, A$ is orthogonal to every row of A. Thus, $\text{Nul}\, A$ and $\text{Row}\, A$ are orthogonal subspaces of $\mathbb{R}^n$.



Every vector $\vec{y} \in \text{Nul}\, A^T$ is orthogonal to every column of A. Thus, $\text{Nul}\, A^T$ and $\text{Col}\, A$ are orthogonal subspaces of $\mathbb{R}^m$.



Example

Prove that every vector $\vec{y} \in \text{Nul } A^T$ is orthogonal to every $\vec{b} \in \text{Col } A$.

Prove that every vector $\vec{x} \in \text{Nul } A$ is orthogonal to every $\vec{b} \in \text{Row } A$.

The Fundamental Theorem of Linear Algebra

Let A be an $m \times n$ matrix with rank r.

- $\dim(\operatorname{Col} A) = r$
- $\dim(\operatorname{Row} A) = r$
- $\dim(\operatorname{Nul} A) = n - r$
- $\dim(\operatorname{Nul} A^T) = m - r$
- $(\operatorname{Nul} A)^\perp = \operatorname{Row} A$
- $\left(\operatorname{Nul} A^T\right)^\perp = \operatorname{Col} A$