NYU

Linear Algebra

Lecture 10

Independence, Basis and Dimension

Dr. Ralph Chikhany

Recap $A \in \mathbb{R}^{m \times n}$ $Ax = b$, n Variable, m Eq.

rank of Matrix = # pivot Variable (in REF)

Free Variable = n − #pivot Variable = $n - r$.

1. if rank = n full column rank matrix.

all the Solution is

$\vec{x} = \vec{x}_{special} + \underbrace{\vec{x}_{null}}$

$\vec{x}_{null} \in Nul(A)$

Free Variables = n − rank = 0

$Nul(A) = \{\vec{0}\}$ and $Ax = b$ can 0 Solution / 1 solution.

2. if rank < n

Free Variable = n − rank $\neq 0$

Free Variable $\underline{ex}$. x_2, x_4 → Solve all the pivot Variable (by x_2, x_4)

$Nul(A) \neq \{\vec{0}\}$ and $Ax = b$ can 0 Solution / inf Solutions

3. if rank = m.   full row rank Matrix

(Today)

★ $Col(A) = \mathbb{R}^m$. which means. $Ax=b$ alway have solution (1 / ∞) for all $b \in \mathbb{R}^m$

Basis of a Vector Space

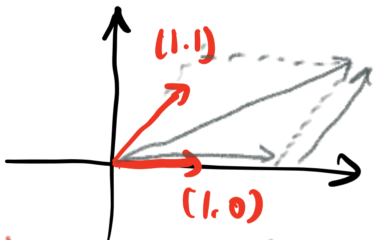
What is a Basis?

A basis β for a vector space V is a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, such that

(1) $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent, and

(2) $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ span V. $\quad \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = V$

Example Find the basis of $\mathbb{R}^2$



1) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ $\quad \begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$

2) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ × $\quad \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}$

$b = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$

3) $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ × $\quad \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\} \neq \mathbb{R}^2$

4) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$ ✓

5) $\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\2\end{bmatrix}\right\}$ ×

What is the basis of $\mathbb{R}^n$

$Ax = b$ always have a solution

- Column vectors of an invertible matrix

n vectors in $\mathbb{R}^n$ and they are linear independent.

Examples

A basis β for a vector space V is a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, such that

(1) $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, and

(2) $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span V.

- standard basis for $\mathbb{R}^2$ is $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

- standard basis for $\mathbb{R}^3$ is $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Column Vectors of I_n

$\vdots$

- standard basis for $\mathbb{R}^n$ is $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$

Examples

A basis β for a vector space V is a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, such that

(1) $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, and

(2) $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span V.

$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ is invertible!

- another basis for $\mathbb{R}^2$ is $\beta = \underline{\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}}$

- the pivot columns form a basis for $\operatorname{Col} A$

- the nullspace solutions form a basis for $\operatorname{Nul} A$

Vector as a Linear Combination of Basis Vectors

Theorem: If $\vec{v} \in V$, then there is a unique way to write $\vec{v}$ as a linear combination of the basis vectors of V.

$\text{span}\{\vec{v}_1, \dots, \vec{v}_n\} = V$, $\quad \vec{v} \in V$, $\quad \vec{v} = \underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n}_{\text{Equation have a unique Solution}}$

__prof__, if

$\vec{v} = c_1\vec{v}_1 + \dots + c_n\vec{v}_n \quad (1)$

$\vec{v} = d_1\vec{v}_1 + \dots + d_n\vec{v}_n \quad (2)$

we want to prof $c_1 = d_1, \dots, c_n = d_n$

$(1) - (2)$: $\quad \vec{0} = (c_1 - d_1)\vec{v}_1 + \dots + (c_n - d_n)\vec{v}_n$

by defn of basis. $\vec{v}_1 \dots \vec{v}_n$ are linear independent.

$c_1 - d_1 \dots c_n - d_n$ should be the trivial Solution

$$\begin{array}{ccc} c_1 - d_1 & & c_n - d_n \\ \| & & \| \\ 0 & & 0 \\ \Downarrow & & \Downarrow \\ c_1 = d_1 & & c_n = d_n \ (!) \end{array}$$

Example

Nul space.

span {all row vectors}

Find bases for the column and row spaces of $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix}$.

row(A) = Col(A^T)

Use REF!

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \begin{matrix} R2-R1 \\ R3-3R1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} R3+R1 \sim \begin{array}{ccc} x_1 & x_2 & x_3 \\ \end{array} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

free

REF

$R_2 = (R_2 + cR_1) - cR_1$

$\text{span}(R_1, R_2) = \text{span}(R_1, R_2 + cR_1)$

$(R_1, R2) \to (R_1, R_2 + cR_1)$

1) find the Row space.

Row operation will not change my row space.

only true for Row

Row(A) = Row(REF)

basis: $\left\{ \begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} 0 \\ 2 \\ 1 \end{smallmatrix} \right\}$

rank(A) = 2 = dim(Row(A))

2) find the Nul Space.

Nul(A) = Nul(REF)

$Ax = 0 \iff REF\, x = 0$ (This is Elimination)

x_3 is free

$x_1 = -x_2 = \frac{1}{2}x_3$

$2x_2 + x_3 = 0 \Rightarrow x_2 = -\frac{1}{2}x_3$

$$= \left\{ \begin{pmatrix} \frac{1}{2}x_3 \\ -\frac{1}{2}x_3 \\ x_3 \end{pmatrix} \middle| \, x_3 \in \mathbb{R} \right\}$$

$\leftarrow$ basis is $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$

$= x_3 \cdot \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \end{pmatrix}$

dim(Nul(A)) = 1 = n - r
= 3 - 2

Example

Nul space.

span{all row vectors}

Find bases for the column and row spaces of $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix}$.   row(A) = Col($A^T$)

Use REF!

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 3 & 1 & -1 \end{bmatrix} \begin{matrix} R2-R1 \\ R3-3R1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} R3+R2 \sim \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

free

REF

dim(Col(A)) = 2

= rank(A)

3) Compute the Column Space   Row operation will change your column using the same way

Ex.   free   pivot   pivot.   REF | A

$$\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{third}} = -\frac{1}{2} \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{first}} + \frac{1}{2} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}_{\text{second}} \qquad \Bigg| \qquad \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{\text{third}} = -\frac{1}{2} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}}_{\text{first}} + \frac{1}{2} \underbrace{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}}_{\text{second}}$$

linear combination Relation will not change

may be linear dependent

Col(A) = span of all column

= span of all **pivot** ← basis

REF: Col(REF) basis $\left\{ \underset{\text{first}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\text{second}}{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}} \right\}$ | A: Col(A). basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$

Examples

Find a basis for $\mathbb{M}_{2\times 2}$, the vector space of all 2×2 matrices.

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \qquad + b \qquad + c \qquad + d$$

Find a basis for the vector space of all 3×3 diagonal matrices.

$$\left\{ \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\} \to \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

basis of

free

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a + b + c + d = 0 \right\} \qquad \{ x \mid \text{condition} \}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

Dimension of a Vector Space

Meaning of Dimension

The dimension of a vector space V is the number of vectors in a basis β for V.

- $\dim(\mathbb{R}^n) = n$
- For an $m \times n$ matrix A with $\operatorname{rank}(A) = r$. → r pivot
  - $\dim(\operatorname{Col} A) = r$
  - $\dim(\operatorname{Row} A) = r$
  - $\dim(\operatorname{Nul} A) = n - r$

dim (col (A)) = dim (row (A))

#Free Variable = n-r

Theorem and Proof Outline

If $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{w}_1, \ldots, \vec{w}_n$ are basis for a vector space V, then $m = n$.

Theorem and Full Proof (Optional Reading)

If $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{w}_1, \ldots, \vec{w}_n$ are basis for a vector space V, then $m = n$.

Suppose $n > m$

Then $\beta_v = \{\vec{v}_1, \ldots, \vec{v}_m\}$ is a basis. Then for $\vec{w}_1 \in V$ we have $\vec{w}_1 = a_{11}\vec{v}_1 + a_{21}\vec{v}_2 + \cdots + a_{m1}\vec{v}_m$. Similarly, for $\vec{w}_2 \in V$ we have $\vec{w}_2 = a_{12}\vec{v}_1 + a_{22}\vec{v}_2 + \cdots + a_{m2}\vec{v}_m$ In general, any $\vec{\omega}_i$ can be written as a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$.

$$\underbrace{[\vec{w}_1 \vec{w}_2 \ldots \vec{w}_n]}_{W} = \underbrace{[\vec{v}_1 \vec{v}_2 \ldots \vec{v}_m]}_{V} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Notice that the matrix (call it A) is short and wide since we assumed $n > m$. Thus $A\vec{x} = \vec{0}$ has a nonzero solution.

\# Vec \# Eq

$$A\vec{x} = \vec{0} \Rightarrow VA\vec{x} = \vec{0} \Rightarrow W\vec{x} = \vec{0}$$

The columns of W are not linearly independent , they can't form a basis (contradiction with the initial assumption)

Suppose $m > n$.

Repeat the same steps and eventually we have:

$$\underbrace{[\vec{v}_1 \vec{v}_2 \ldots \vec{v}_n]}_{V} = \underbrace{[\vec{w}_1 \vec{w}_2 \ldots \vec{w}_m]}_{W} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{m2} & \cdots & b_{nm} \end{bmatrix}$$

Notice that the matrix (call it B) is short and wide since we assumed $m > n$. Thus $B\vec{x} = \vec{0}$ has a nonzero solution.

$$B\vec{x} = \vec{0} \Rightarrow WB\vec{x} = \vec{0} \Rightarrow V\vec{x} = \vec{0}$$

The columns of V are not linearly independent , they can't form a basis (contradiction with the initial assumption)

Conclusion: The only way to avoid these contradictions is to have $m = n$.

Row Space of a Matrix

Example

Find a basis and the dimension of Col A and ~~Nul~~ row A, where

rank(A) = 3

$$A = \begin{bmatrix} 1 & -3 & -6 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

$R2 - 5R_1$

REF

using Elimination.

$$A \sim \begin{bmatrix} 1 & -3 & -6 & 0 \\ 0 & 15 & 30 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \xrightarrow{R3 - \frac{1}{15}R2} \sim \begin{bmatrix} 1 & -3 & -6 & 0 \\ 0 & 15 & 30 & 4 \\ 0 & 0 & 0 & -19/15 \\ 0 & 0 & 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -6 & 0 \\ 0 & 15 & 30 & 4 \\ 0 & 0 & 0 & -19/15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

← pivot row
← pivot row
← pivot row

① while doing the row operation, the row space is not changing !!
the column space will change!

row(A) = row(REF) = span $\left\{ \begin{pmatrix} 1 \\ -3 \\ -6 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 15 \\ 30 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -19/15 \end{pmatrix} \right\}$ ← basis !!

pivot row

dim(row(A)) = 3
= rank (A)

② Calculate the Column space. | linear combination relation still hold.

always holds

Ex. in REF $\begin{bmatrix} -6 \\ 30 \\ 0 \\ 0 \end{bmatrix} = 2 \times \begin{bmatrix} -3 \\ 15 \\ 0 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ in A $\begin{bmatrix} -6 \\ 0 \\ 2 \\ 0 \end{bmatrix} = 2 \times \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 0 \times \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}$

Example

Find a basis and the dimension of Col A and Nul A, where

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$$\begin{bmatrix} 1 & -3 & -6 & 0 \\ 0 & 15 & 30 & 4 \\ 0 & 0 & 0 & -19/15 \\ 0 & 0 & 0 & 5 \end{bmatrix} \text{REF.}$$

$$A = \begin{bmatrix} 1 & -3 & -6 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

$\dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A))$

$= \mathrm{rank} = \#\,\mathrm{pivots}$!

pivot Columns of REF — Go back

basis of $\mathrm{Col}(A) = \begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 4 \\ -1 \\ 5 \end{pmatrix}$

!! Go back to matrix A to select Column Vector!!

$\dim(\mathrm{Col}(A)) = 3 = \mathrm{rank}(A)$

③ Calculate the null space.

$\mathrm{Nul}(A) = \mathrm{Nul}(\mathrm{REF})$

$x_4 = 0$, x_3 is the free variable, $x_2 = -2x_3$, $x_1 = 3x_2 + 6x_3 = 0$

$\Rightarrow \mathrm{Nul}(\mathrm{REF}) = \mathrm{span}\left\{ \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}$

$\begin{pmatrix} 0 \\ -2x_3 \\ x_3 \\ 0 \end{pmatrix} = x_3 \cdot \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}$

$n = 4$

$\dim[\mathrm{Nul}(A)] = 4 - 3$

$= 1$

Lecture 11

The Four Fundamental Subspaces

Dr. Ralph Chikhany

Strang Sections 3.5 – Dimensions of the Four Subspaces

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed) and N. Hammoud's NYU lecture notes.

Thm. $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.

$$\begin{array}{ccc} \| & & \| \\ \dim(\mathrm{Col}(A)) & = & \dim(\mathrm{row}(A^T)) \\ \| & & \| \\ \dim(\mathrm{row}(A)) & = & \dim(\mathrm{Col}(A^T)) \end{array}$$

$\mathrm{rank}(A) \geq \mathrm{rank}(A \cdot B)$

$\mathrm{rank}(B) \geq \mathrm{rank}(A \cdot B)$

$\mathrm{Col}(A) = \mathrm{row}(A^T)$

$\mathrm{row}(A) = \mathrm{col}(A^T)$

$\mathrm{rank}(A) = \dim(\mathrm{Col}(A))$ $\qquad$ $\mathrm{rank}(AB) = \dim(\mathrm{Col}(AB))$

Claim. $\mathrm{Col}(AB) \subseteq \mathrm{Col}(A)$ (is a subspace) ??

by Matrix × Matrix:

$$AB = A\begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_p \end{bmatrix} = \begin{bmatrix} A\vec{c}_1 & \cdots & A\vec{c}_p \end{bmatrix}$$

$\uparrow$ Column vectors of B $\qquad$ $A\vec{c}_1 \in \mathrm{Col}(A)$ $\qquad$ $A\vec{c}_p \in \mathrm{Col}(A)$

All the Column vectors of AB, lies in $\mathrm{Col}(A)$

$\Rightarrow \mathrm{Col}(AB) \subseteq \mathrm{Col}(A)$

$$\begin{array}{ccc} \dim(\mathrm{Col}(AB)) & \leq & \dim(\mathrm{Col}(A)) \\ \| & & \| \\ \mathrm{rank}(AB) & & \mathrm{rank}(A) \end{array}$$

Matrix Subspaces

The Subspaces Associated with a Matrix

$\mathbb{R}^{m\times n}$

Consider an $m \times n$ matrix $A = [\vec{a}_1 \quad \vec{a}_2 \quad \dots \quad \vec{a}_n]$, where $\vec{a}_i \in \mathbb{R}^m$ $(1 \le i \le n)$.
Then, the four fundamental subspaces associated with A are:

(1) The column space Col A=span{pivot columns}$\subset \mathbb{R}^m$

(2) The row space Row $A =$ Col $A^T \subset \mathbb{R}^n$

both $\mathbb{R}^m$

(3) The nullspace Nul $A = \left\{\vec{x} \middle| A\vec{x} = \vec{0}\right\} \subset \mathbb{R}^n$

$\underset{m\times n}{A}\ \underset{n}{\vec{x}}$ $\mathbb{R}^n$

(4) The left nullspace Nul $A^T = \left\{\vec{y} \middle| A^T\vec{y} = \vec{0}\right\} \subset \mathbb{R}^m$

$\underset{n\times m}{A^T}\ \underset{m}{y}$

The Subspaces Associated with a Matrix

Consider an $m \times n$ matrix $A = [\vec{a}_1 \quad \vec{a}_2 \quad \dots \quad \vec{a}_n]$, where $\vec{a}_i \in \mathbb{R}^m$ $(1 \le i \le n)$.
Then, the four fundamental subspaces associated with A are:

(1) The column space Col A=span{pivot columns}$\subset \mathbb{R}^m$

(2) The row space Row $A = \text{Col } A^T \subset \mathbb{R}^n$

(3) The nullspace Nul $A = \left\{\vec{x} \middle| A\vec{x} = \vec{0}\right\} \subset \mathbb{R}^n$

(4) The left nullspace Nul $A^T = \left\{\vec{y} \middle| A^T\vec{y} = \vec{0}\right\} \subset \mathbb{R}^m$

If A is and $m \times n$ matrix with rank r, then

(1) $\dim(\text{Col } A) = r$

(2) $\dim(\text{Row } A) = r$

$Ax=b$. m Eq n Var. # Free Var $n-r$

(3) $\dim(\text{Nul } A) = n - r$

$A^Ty=b$ n Eq m Var # Free Va $m-r$

(4) $\dim(\text{Nul } A^T) = m - r$

Col(A), Nul $(A^T) \subseteq \mathbb{R}^m$ $\quad$ dim (Nul (A^T)) + dim (Col (A)) = m

Row(A), Nul (A) $\subseteq \mathbb{R}^n$ $\quad$ dim (Nul (A)) + dim (Row(A)) = n

$A\vec{x} = \vec{0}$ $\quad$ $\vec{r}_i$ are row vectors

$$A = \begin{bmatrix} \vec{r}_1^{\,T} \\ \vec{r}_2^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{bmatrix} \qquad \begin{bmatrix} \vec{r}_1^{\,T} \\ \vec{r}_2^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{bmatrix}\vec{x} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix}$$

$A\vec{x} = \vec{0}$ means $\vec{r}_1 \cdot \vec{x} = \vec{r}_2 \cdot \vec{x} = \cdots = \vec{r}_m \cdot \vec{x} = 0$

$\vec{r}_1 \perp \vec{x}$ $\qquad$ $\vec{r}_2 \perp \vec{x}$ $\qquad$ $\vec{r}_m \perp \vec{x}$

Nul(A) is all the vector $\vec{x}$ that is orthogonal to all the row vectors of A $(\vec{r}_1 \cdots \vec{r}_n)$

row space

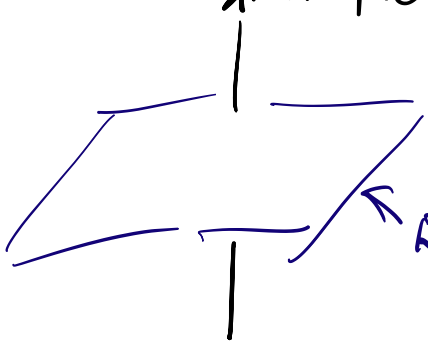
Nul(A^T) is all the vector $\vec{x}$ that is orthogonal to all the column vectors of A

Column space

in $\mathbb{R}^3$, row space is a plane $\quad$ dim(row) = 2

dim(Nul) = dim(all vectors orthogonal to row space)

= 3 − 2 = 1



Null space

Row space

Example

Find a basis and dimension of the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix}.$$

Example

Find a basis and dimension of the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix}.$$

Example

Find a basis and dimension of the four fundamental subspaces of

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix}.$$

Example

Find a basis and dimension of the four fundamental subspaces of

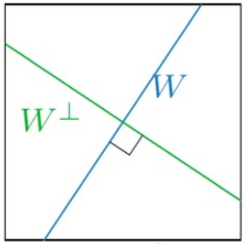
$$A = \begin{bmatrix} 1 & 3 & 5 & 0 & 7 \\ 0 & 0 & 0 & 1 & 2 \\ 1 & 3 & 5 & 1 & 9 \end{bmatrix}.$$

Orthogonality

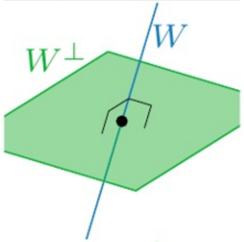
Orthogonal Subspaces

Two subspaces V and W of a vector space are orthogonal if every vector $\vec{v} \in V$ is orthogonal to every vector $\vec{w} \in W$. That is,

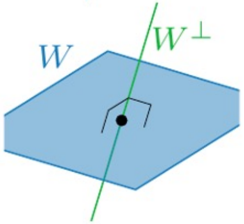
$$\vec{v}^T \vec{w} = 0 \quad \text{for all } \vec{v} \in V \text{ and all } \vec{w} \in W.$$



Every vector $\vec{x} \in \operatorname{Nul} A$ is orthogonal to every row of A. Thus, $\operatorname{Nul} A$ and $\operatorname{Row} A$ are orthogonal subspaces of $\mathbb{R}^n$.



Every vector $\vec{y} \in \operatorname{Nul} A^T$ is orthogonal to every column of A. Thus, $\operatorname{Nul} A^T$ and $\operatorname{Col} A$ are orthogonal subspaces of $\mathbb{R}^m$.



The Fundamental Theorem of Linear Algebra

Let A be an $m \times n$ matrix with rank r.

- $\dim(\mathrm{Col}\, A) = r$
- $\dim(\mathrm{Row}\, A) = r$
- $\dim(\mathrm{Nul}\, A) = n - r$
- $\dim(\mathrm{Nul}\, A^T) = m - r$
- $(\mathrm{Nul}\, A)^\perp = \mathrm{Row}\, A$
- $\left(\mathrm{Nul}\, A^T\right)^\perp = \mathrm{Col}\, A$