NYU

Linear Algebra

Lecture 8

Nullspace

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Recap

- Vector Space V (VS1 – VS8)

if $\vec{v}_1, \vec{v}_2 \in V$, then $c_1\vec{v}_1 + c_2\vec{v}_2 \in V$ Closed respect to linear Combination

Example.

$\{x \mid Ax = 0\}$ ✓ Null space $Nul(A)$

$\{x \mid Ax = b\}$ $(b \neq 0)$ ✗

$Span\{\vec{v}_1 \cdots \vec{v}_n\}$ ✓ Column Space $Col(A)$

$A = [\vec{v}_1, \cdots, \vec{v}_n]$ $\vec{v}_1 \cdots \vec{v}_n$ are Column Vectors.

Why $Col(A)$? $b \in Col(A) \Leftrightarrow Ax = b$ have a Solution (1 solution / infine Solut...)

Why $Nul(A)$? infinite Solution

$Ax = b$ Now a special Solution x_{spec}. $Ax_{spec} = b$

If x is another solution of $Ax = b$ Then $x = x_{spec} + \underbrace{x_n}_{x_n \in Nul(A)}$

Check $x - x_{spec} \in Nul(A)$. $A(x - x_{spec}) = Ax - Ax_{spec} = b - b = \vec{0}$

$\Rightarrow x - x_{spec} \in Nul(A)$

Q If $Ax = b$ have a unique Solution. What is the $Nul(A)$? $\{\vec{0}\}$

Strang Sections 3.2 – The nullspace of A

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

The Nullspace of a Matrix

Nullspace

Let A be an $m \times n$ matrix, such that $A = [\vec{a}_1 \ \vec{a}_2 \ \ldots \ \vec{a}_n]$, where $\vec{a}_i \in \mathbb{R}^m$ $(1 \le i \le n)$. The nullspace of A consists of all solutions $\vec{x}$ to $A\vec{x} = \vec{0}$. That is,

$$\text{Nul}\, A = \left\{ \vec{x} \,\middle|\, A\vec{x} = \vec{0} \right\}.$$

$$\text{Nul}(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \,\middle|\, \begin{matrix} x_1 = -x_3 \\ x_2 = 0 \\ x_4 = 0 \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} -x_3 \\ 0 \\ x_3 \\ 0 \end{pmatrix} \,\middle|\, x_3 \in \mathbb{R} \right\}$$

Compute the nullspace of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 2 \\ 1 & 3 & 1 & 2 \end{bmatrix}$.

Use Elimination Method

$Ax = 0$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 1 & 3 & 1 & 2 & 0 \end{array}\right) \begin{matrix} \\ R_2 + R_1 \\ R_3 - R_1 \end{matrix} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 & 0 \end{array}\right) \quad R_3 - 2R_2$$

REF!

$$\rightarrow \boxed{\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)}$$

$\leftarrow$ to determine x_1 $\quad x_1 + x_2 + x_3 - x_4 = 0$. $\quad x_1 = -x_3$

$\leftarrow$ to determine x_2. $\quad x_2 + x_4 = 0$. $\quad x_2 = 0$

$\leftarrow$ I don't have pivot in third Column So x_3 can be any number

free variable

$\leftarrow$ to determine $x_4 = 0$

Nullspace

Let A be an $m \times n$ matrix, such that $A = [\vec{a}_1\ \vec{a}_2\ \ldots\ \vec{a}_n]$, where $\vec{a}_i \in \mathbb{R}^m$ $(1 \le i \le n)$. The nullspace of A consists of all solutions $\vec{x}$ to $A\vec{x} = \vec{0}$. That is,

$$\text{Nul}\,A = \left\{ \vec{x} \,\middle|\, A\vec{x} = \vec{0} \right\}.$$

Compute the nullspace of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 2 \\ 1 & 3 & 1 & 2 \end{bmatrix}$.

(can be any number)

x_3 is free

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 1 & 3 & 1 & 2 & 0 \end{array}\right] \begin{matrix} R_2 + R_1 \\ R_3 - R_1 \end{matrix} \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 & 0 \end{array}\right] R3 - 2R2 \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

determine x_1 First Eq — determin x_2 Second Eq — to determine x_4 Third Eq

R3: $x_4 = 0$

R2: $x_2 + x_4 = 0 \Rightarrow x_2 = -x_4 = 0$

R1: $x_1 + x_2 + x_3 - x_4 = 0 \Rightarrow x_1 = x_4 - x_3 - x_2 = -x_3$

$$\text{Nul}(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \,\middle|\, \begin{matrix} x_4 = x_2 = 0 \\ x_1 = -x_3,\ x_3 \in \mathbb{R} \end{matrix} \right\} = \left\{ \begin{bmatrix} -x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} \,\middle|\, x_3 \in \mathbb{R} \right\}$$

Theorem

The nullspace of A is a subspace of $\mathbb{R}^n$.

Row Echelon Form (REF)

REF

An $m \times n$ matrix is in echelon or row-echelon form (REF) if:

(1) All rows consisting entirely of zeros lie beneath all nonzero rows.

(2) The first nonzero element in any row is called a **pivot**.

(3) Any pivot must lie to the right of any pivot above it.

"generalization to upper Triangle"

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

✓

×
by (1)
0, 0, 0 should at bottom!

×
by (2)
The second pivot × lies to the right of first one

✓

Pivot Columns vs. Free Columns

Pivot Columns: The first pivot column in any matrix A is the left-most nonzero column. The top element in that column is called the first *pivot* or *pivot position*. If the first pivot is zero, we swap rows to get a nonzero pivot. A pivot column in a matrix in REF is a column that contains *exactly* one pivot.

no pivot!

Free Columns: A free column in a matrix in REF is a column that contains *no* pivots.

pivot columns linear independent

free columns lies in the span of pivot columns!

free Variable

$$\mathrm{Nul}(A) = \left\{ \begin{pmatrix} 0 \\ x_2 \\ x_4 \\ x_4 \end{pmatrix} \middle| \; x_2 . x_4 \in \mathbb{R} \right\}$$

rank = 3 rank = 2 rank = 2

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{c} \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} & x_2 & & x_4 \end{array} \end{array} \begin{array}{l} \leftarrow x_1 \text{ determine} \\ \leftarrow x_3 \text{ determine} \\ \end{array}$$

$x_1 = x_2 - x_4 = x_4 - x_4 = 0$

$x_3 = x_4$

pivot col free free free

rank 2

x_2, x_4 are free variable!

Pivot Columns vs. Free Columns

→ can solve by the pivot Rows

Pivot Columns: The first pivot column in any matrix A is the left-most nonzero column. The top element in that column is called the first *pivot* or *pivot position*. If the first pivot is zero, we swap rows to get a nonzero pivot. A pivot column in a matrix in REF is a column that contains *exactly* one pivot.

→ free Variable

Free Columns: A free column in a matrix in REF is a column that contains *no* pivots.

① Pivot Columns are linear inde...

② free Columns ∈ span {pivot Columns}

→ Col(A) = span { pivot Columns }

Free Columns!

Example of ①
⇒

rank = 3 $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ rank = 2 $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ rank = 2 $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ rank = 2 $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

②

pivot Column

$c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}2\\4\\0\end{bmatrix} + c_3\begin{bmatrix}3\\2\\3\end{bmatrix} = \vec{0}$ ← $c_1 = 0$ ← $c_2 = 0$ ← $c_3 = 0$

$c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}-1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\-1\\0\end{bmatrix}$ ← solve $c_1 = c_2 + 1 = 0$ ← Solve $c_2 = -1$ ← always hold

Example

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 7 \\ 2 & 5 & 9 \end{bmatrix}$. Write A in row-echelon form and find its nullspace.

$A\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 3 & 6 & 7 & 0 \\ 2 & 5 & 9 & 0 \end{array}\right] \begin{array}{l} R2-3R1 \\ R3-2R1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 1 & 7 & 0 \end{array}\right]$$

!!

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \begin{array}{l} R2 \\ R3 \end{array}$$

Null space is not Changing !

$\rightarrow x_1 = 0$

$\rightarrow x_2 = 0$

$\rightarrow x_3 = 0$

$\text{Null}(A) = \{\vec{0}\}$

Reduced Row Echelon Form (RREF)

RREF

To find the reduced row-echelon form (RREF) of a matrix A:

(1) First find the REF of A. That is, find the *pivots* and use them to make all elements *below* them equal zero.

(2) Then, use the obtained *pivots* to make all elements *above* them equal zero.

(3) Lastly, make all pivots equal 1.

This procedure does not change the nullspace of the original matrix A.

Example

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 7 \\ 2 & 5 & 9 \end{bmatrix}$. Write A in reduced row-echelon form and find its nullspace.

Rank

What is the Rank of a Matrix?

The rank of an $m \times n$ matrix A is the number of pivots. We call this number r.

REF → count pivots

We think of the rank as the "true size" of a matrix. For example, the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ is 2×3, however, its REF is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. That is, the second row is not giving any additional information; it is just a copy of the first row.

free col ∈ span{pivot columns}

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 7 \\ 2 & 5 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ & 1 & 7 \\ & & 4 \end{bmatrix} \quad \text{rank} = 3$$

rank 3

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

row. #Eq

free Column

Column # Variable

matrix $A \in \mathbb{R}^{m \times n}$, rank r ← # pivot column

$r \leq m$, $r \leq n$ $\qquad$ $n-r$

what is possible largest r. and what is # of free variable?

What is the Rank of a Matrix?

The rank of an $m \times n$ matrix A is the number of pivots. We call this number r.

We think of the rank as the "true size" of a matrix. For example, the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ is 2×3, however, its REF is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. That is, the second row is not giving any additional information; it is just a copy of the first row.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 7 \\ 2 & 5 & 9 \end{bmatrix}$$

REF

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

rank 3

free variable

rank = 3

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

4 Eq and 6 variable system.

rank = 3 → 6-3=3 free variable

3 pivot variable can be solved

I can't solve / can be any number

Lecture 9
Complete Solutions

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Strang Sections 3.3 – The Complete Solutions to $Ax = B$

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Complete Solutions

Theorem

\# Rows, # Eq   # Col, # Variable   # Eq < # Variable

Let A be an $m \times n$ matrix such that $m < n$. In this case, we are guaranteed to have free columns, and the system $A\vec{x} = \vec{b}$ will have more unknowns than equations, so it will have free variables associated with the free columns. Thus, this system will always have either an *infinite number* of solutions or *no* solutions.

rank $\le m < n$, number of free Variable = n − rank > 0

Let's look at this through an example.

$\text{Nul}(A) \neq \{\vec{0}\}$

Example

Solve $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}$.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 2 & 4 & 4 & 8 & 2 \\ 4 & 8 & 6 & 8 & 10 \end{array}\right] \quad \begin{array}{l} R2 - 2R1 \\ R3 - 4R1 \end{array}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 2 & 8 & -6 \\ 0 & 0 & 2 & 8 & -6 \end{array}\right] \quad R3 - R2 \quad \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$= -2x_2 + 4x_4 = 7$

$= 4 - (-3 - 4x_4) - 2x_2$

$x_1 = 4 - x_3 - 2x_2$

$x_3 = -3 - 4x_4$

x_2, x_4 are free variables!

x_1, x_3 are pivot variable!

special solution (put. $x_2 = x_4 = 0$) Null space!

The Complet Solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2 + 4x_4 + 7 \\ x_2 \\ -4x_4 - 3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ -3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ 0 \\ -4 \\ 1 \end{pmatrix}$$

Example

Solve $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}$.

4 Variable, 3 Equations

rank(A) = 2 , 4 − 2 Free Variables !!

$A \in \mathbb{R}^{4\times 3}$, what is the largest possible rank? 3

$A \in \mathbb{R}^{3\times 4}$, what is the largest possible rank? 3

rank $\leq m$. rank $\leq n$

#row #Columns

Particular vs. Nullspace Solution

- The particular solution is obtained by setting the free variables to zero. The particular solution solves $A\vec{x}_p = \vec{b}$. special solution (by settig free var to 0)

- The nullspace solution is obtained by setting the right-hand-side, i.e., the vector $\vec{b}$, to $\vec{0}$. There are $n - r$ nullspace solutions which solve $A\vec{x}_n = \vec{0}$.
  #Free variable

- The complete solution to $A\vec{x} = \vec{b}$ can be expressed as $\vec{x} = \vec{x}_p + \vec{x}_n$.
  Special Null space!

More on Ranks and Systems

Rank

Recall: the rank r of a matrix A is the number of pivots.

- The r pivot columns are linearly independent.
- There are $n - r$ free columns.
- Since Col A=span{pivot columns}, then the column space spans an r-dimensional space.
- The "dimension" of the nullspace is $n - r$.

Full Column Rank $r = n$

if $r = n$, then $m \geq n$! # Rows > # Cols "tall" matrix

Eq > # Variabl

Suppose A is an $m \times n$ matrix. Then A has full column rank $r = n$ if:

- all columns of A are pivot columns
- all columns of A are linearly independent
- there are no free columns $\Longrightarrow$ no free solutions
- $\text{Nul}\, A = \{\vec{0}\}$ # Free Variables $= n - r = 0$
- if $A\vec{x} = \vec{b}$ has a solution, then it has exactly one solution.

Full Row Rank $r = m$

Because $r = m$ $\Rightarrow$ $m \le n$ #row $\le$ # Column   "Fat" Matrix

$r \le n$ # Eq $\le$ # Variable

Suppose A is an $m \times n$ matrix. Then A has full row rank $r = m$ if:

REF $\begin{bmatrix} 1 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \end{bmatrix}$

- all rows of A have pivot positions
- all rows of A are linearly independent (try)
- there are $n - r = n - m$ nullspace solutions
- The column space of A spans all of $\mathbb{R}^m$
- $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$

$\operatorname{Col}(A) = \mathbb{R}^m$

$\parallel$

span{Pivot Column}

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} * \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} * \\ * \\ 1 \end{pmatrix} \begin{matrix} \leftarrow c_1 \\ \leftarrow c_2 \\ \leftarrow c_3 \end{matrix}$$

$Ax = b$

$A \in \mathbb{R}^{m \times n}$

① $m < n \Rightarrow \begin{cases} 0 & \text{Solution} \\ +\infty & \text{Solutions} \end{cases}$

rank $\leq m < n$, # Free Variables $= n -$ rank > 0

② $m = n$ A is a square matrix

1. rank $= m = n$ A is full Column rank

   1. # Free $= n -$ rank $= 0 \Rightarrow$ Nul (A) $= \{\vec{0}\}$

   2. A is also full row rank (rank $= m$)

      ① Col (A) $= \mathbb{R}^m$

      and $Ax = b$ always have a Solution

   $Ax = b$ have and only have single Solution.

   ☆ A has a inverse Matrix !!!

   - rows are linear indepedent ← row full rank
   - Cols are linear independent ← Col full rank

   every Col is a pivot Col

2. rank $< m = n$

   # Free $= n -$ rank $> 0 \Rightarrow \begin{cases} 0 & \text{Solutions} \\ \infty & \text{Solutions} \end{cases}$

#Eq #Variable

③ $m > n$

1. largest possible rank $= n$

   # Free Variable $= n -$ rank $= 0$

   $\begin{cases} 0 & \text{Solutions} \quad b \notin \text{Col}(A) \\ 1 & \text{Solution} \quad b \in \text{Col}(A) \end{cases}$

2. rank $< n < m$

   # Free Variables $= n -$ rank > 0

   $\begin{cases} 0 & \text{Solution} \quad b \notin \text{Col}(A) \\ +\infty & \text{Solutions} \quad b \in \text{Col}(A) \end{cases}$

Whether $Ax = b$ hase Solution should we $b \in$ Col (A)

Rank and Solvability

Let A be an $m \times n$ matrix with rank r. The solutions to $A\vec{x} = \vec{b}$ can be classified as follows:

(1) $r = m,\ r = n \implies$ square invertible $\implies A\vec{x} = \vec{b}$ has 1 solution

(2) $r = m,\ r < n \implies$ short and wide $\implies A\vec{x} = \vec{b}$ has an infinite number of solutions

(3) $r < m,\ r = n \implies$ tall and thin $\implies A\vec{x} = \vec{b}$ has 0 or 1 solution

(4) $r < m,\ r < n \implies$ not full rank $\implies A\vec{x} = \vec{b}$ has either an infinite number of solutions or no solutions

Rank and Solvability

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