

Lecture 8 Nullspace

Dr. Ralph Chikhany

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Recap
   - Vector Space V (VS) - VSB)
                                                          closed respect to linear Combination
          if $1, $\frac{1}{\sqrt{2}} \in \text{V} , then $C_1 \overline{\sqrt{1}} + C_1 \overline{\sqrt{2}} \in \text{V}
    Example.
           \{x \mid Ax = 0 \} Null space Null(A)
           \{x \mid Ax = b\} (b \neq 0) x
           Span { vi... vn} V Glumn Space GI(A)
              A=[Vi,..., Vn] Vi ... Vn are Glumn Vectors.
   Why GI(A)? be GI(A) (=) Ax=b have a Solution (1 solution / infine solution)
   Why Nu (A)? infinite Solution
       Ax = b Now a special Solution X spec. Axspec = b
      If x is another solution of Ax = b Then X = X \operatorname{spec} + X \operatorname{n}

X \cap E \operatorname{Nul}(A)
Check X - X_{Spec} \in Nul(A). A(X - X_{Spec}) = A_X - AX_{Spec} = b - b = 0
                                 > X- Kipoc € Nul (A)
       D If Ax = b have a unique solution. What is the Mil (A) ?
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Strang Sections 3.2 – The nullspace of A



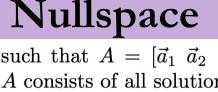
The Nullspace of a Matrix

Flimination Method

llse

Let
$$A$$
 be an $m \times n$ matrix, such that $A = [\vec{a}_1 \ \vec{a}_2]$ $(1 \le i \le n)$. The nullspace of A consists of all solution

Let
$$A$$
 be an $m \times n$ matrix, such that $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n]$, where $\vec{a}_i \in \mathbb{R}^m$ $(1 \le i \le n)$. The nullspace of A consists of all solutions \vec{x} to $A\vec{x} = \vec{0}$. That is,



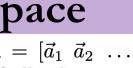
PEF!

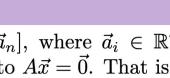
O lo le termine X1 X1 + X2 + X3 - X4 =0. X1 =- X3

to de termine X2. X2 + X4 =0. X3 =0

EI don't have pivot in third Column so X3 can be

to determine X4 =0 free variable only number





$$G_i \in \mathbb{R}^m$$
That is,
 $M = -X_3$

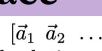
That is,
$$\begin{array}{c}
X_1 = X_2 \\
X_2 = X_3
\end{array}$$

$$\operatorname{Nul} A = \left\{ \vec{x} \mid A\vec{x} = \vec{0} \right\}. \quad \operatorname{Nul} (A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{array}{c} x_1 = -x_3 \\ x_4 \\ x_4 = 0 \end{array} \right\}$$
Compute the nullspace of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 2 \\ 1 & 3 & 1 & 2 \end{bmatrix}.$

$$\bar{x}$$

$$= \left\{ \left(\begin{array}{c} x_3 \\ x_3 \end{array} \right) \mid x_3 \in \mathbb{R} \right\}$$

$$\begin{pmatrix}
1 & 1 & -1 & 0 \\
-1 & 0 & + & 2 & 0 \\
1 & 3 & 1 & 2 & 0
\end{pmatrix}
\begin{pmatrix}
R_{2} + R_{1} \\
R_{3} - R_{1}
\end{pmatrix}
=
\begin{pmatrix}
0 & 1 & 0 & 1 \\
0 & 2 & 0 & 3
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
R_{3} - 2 \\
R_{2}
\end{pmatrix}$$



Nullspace

 $\begin{bmatrix}
1 & 1 & -1 & | & 0 \\
-1 & 0 - 1 & 2 & | & 0 \\
0 & 3 & 1 & 2 & | & 0
\end{bmatrix}
R₂ + R₁
<math display="block">\Rightarrow \begin{bmatrix}
0 & 1 & 0 & | & 0 \\
0 & 2 & 0 & 3 & | & 0
\end{bmatrix}
R₃ - 2R₂
<math display="block">\Rightarrow \begin{bmatrix}
0 & 1 & 0 & | & 0 \\
0 & 2 & 0 & 3 & | & 0
\end{bmatrix}
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 $\operatorname{Nul} A = \left\{ \vec{x} \mid A\vec{x} = \vec{0} \right\}.$

that
$$A = [\vec{a}_1 \ \vec{a}_2]$$

Let A be an $m \times n$ matrix, such that $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n]$, where $\vec{a}_i \in \mathbb{R}^m$

 $X_1 + X_2 + X_3 - X_4 = 0 \Rightarrow X_1 = X_4 - X_2 - X_2 = -X_3$

 $|W|(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{array}{c} x_1 = x_1 \\ x_2 = x_1 \\ x_3 \end{pmatrix} \middle| \begin{array}{c} x_2 \in \mathbb{R} \\ x_3 \\ x_4 = x_2 = x_1 \\ x_3 \in \mathbb{R} \\ x_4 = x_2 = x_1 \\ x_4 = x_2 = x_2 \\ x_5 = x_1 \\ x_5 = x_1 \\ x_5 = x_1 \\ x_5 = x_2 \\ x_5 = x_1 \\ x_5 = x_2 \\ x_5 = x_1 \\ x_5 = x_1 \\ x_5 = x_2 \\ x_5 = x_1 \\ x_5 = x_2 \\ x_5 = x_1 \\ x_5 = x_2 \\ x_5 = x_2 \\ x_5 = x_1 \\ x_5 = x_2 \\ x_5 = x_3 \\ x_5 = x_2 \\ x_5 = x_2 \\ x_5 = x_3 \\ x_5 = x_3 \\ x_5 = x_2 \\ x_5 = x_3 \\ x_5 = x_$

Compute the nullspace of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 2 \\ 1 & 3 & 1 & 2 \end{bmatrix}$.

R2: XL+ XL =0 => XL=-X4=D

R3:

Xn = 0

that
$$A = [\vec{a}_1 \ \vec{a}_2]$$

h that
$$A = [\vec{a}, \vec{a}]$$

 $(1 \le i \le n)$. The nullspace of A consists of all solutions \vec{x} to $A\vec{x} = \vec{0}$. That is,

(can be any number)

Theorem

The nullspace of A is a subspace of \mathbb{R}^n .

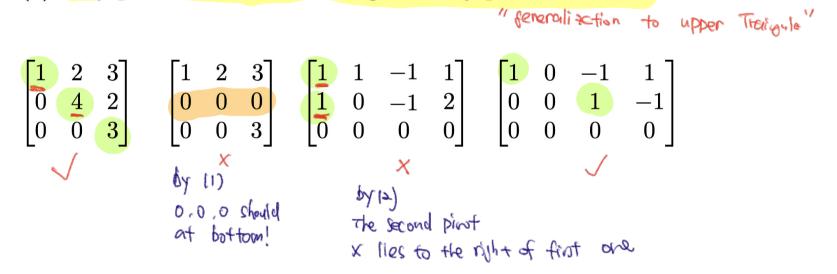


Row Echelon Form (REF)

REF

An $m \times n$ matrix is in echelon or row-echelon form (REF) if:

- (1) All rows consisting entirely of zeros lie beneath all nonzero rows.
- (2) The first nonzero element in any row is called a **pivot**.
- (3) Any pivot must lie to the right of any pivot above it.



Pivot Columns vs. Free Columns

Pivot Columns: The first pivot column in any matrix A is the left-most nonzero column. The top element in that column is called the first pivot or pivot position. If the first pivot is zero, we swap rows to get a nonzero pivot. A pivot column in a matrix in REF is a column that contains exactly one pivot.

Free Columns: A free column in a matrix in REF is a column that contains no pivots.

pivot columns linear independent free columns lies in the span of pivot columns!

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Null(A) =
$$\begin{bmatrix} x_{ij} \\ x_{ij} \end{bmatrix}$$

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-> free Uniate **Free Columns:** A free column in a matrix in REF is a column that contains 1) Pivot Odumus are linear inda--no pivots. Free Glumns! @ free (dums & spen Frank = 3

Tank Groff (I [] = [-1] & solve (I = Cett of Solve (I = Pivo + Column $C_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} 4 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 0 \leftarrow C_2 = 0$

Example

Let
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 7 \\ 2 & 5 & 9 \end{bmatrix}$$
. Write A in row-echelon form and find its nullspace.

$$\begin{bmatrix} 2 & 5 & 9 \end{bmatrix} \qquad Ax = 0$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 6 & 7 & 0 \\ 2 & 7 & 9 \end{bmatrix} \xrightarrow{R} \begin{bmatrix} 2 & 3R \\ R3 - 2R \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 7 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 17 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \xrightarrow{\text{P2}} \text{Null space is not Chargins} \xrightarrow{} \xrightarrow{>} \times_{1} = 0$$

$$\Rightarrow \times_{1} = 0$$

$$\Rightarrow \times_{2} = 0$$



Reduced Row Echelon Form (RREF)

RREF

To find the reduced row-echelon form (RREF) of a matrix A:

- (1) First find the REF of A. That is, find the *pivots* and use them to make all elements below them equal zero.
- (2) Then, use the obtained *pivots* to make all elements *above* them equal zero.
- (3) Lastly, make all pivots equal 1.

This procedure does not change the nullspace of the original matrix A.

Example

Let
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 7 \\ 2 & 5 & 9 \end{bmatrix}$$
. Write A in reduced row-echelon form and find its nullspace.



Rank

What is the Rank of a Matrix?

The rank of an $m \times n$ matrix A is the number of pivots. We call this number r.

We think of the rank as the "true size" of a matrix. For example, the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ is 2×3 , however, its REF is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. That is, the second row is not giving any additional information; it is just a copy of the first row.

free C)
$$\in$$
 span $\{$ Divot Columns $\}$

[1 2 1]
3 6 7
2 5 9]
- [1 7]

Mank = 3

[1 0 3 1 0 1]
0 1 2 -1 0 2
0 0 0 0 4 -6
0 0 0 0 0 0 0

There Column

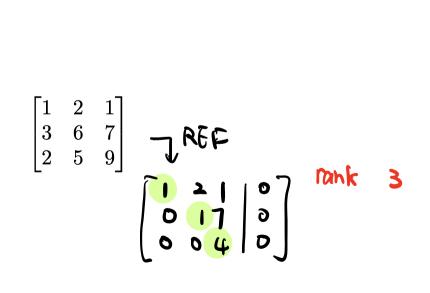
Mothix $A \in \mathbb{R}^{m \times n} \in \text{Column # Knieble}$

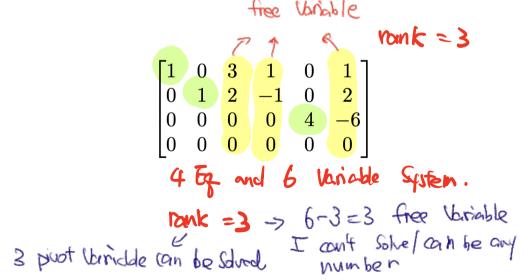
What is possible largest r , and what is $\#$ of free Usrieble?

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Lecture 9 Complete Solutions

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Strang Sections 3.3 – The Complete Solutions to Ax = B



Complete Solutions

Theorem

Tow, by # # Let A be an $m \times n$ matrix such that m < n. In this case, we are guaranteed to

Let A be an $m \times n$ matrix such that m < n. In this case, we are guaranteed to have free columns, and the system $A\vec{x} = \vec{b}$ will have more unknowns than equations, so it will have free variables associated with the free columns. Thus, this system will always have either an *infinite number* of solutions or *no* solutions.

rank
$$\leq m < n$$
, number of free Uniable = $n - rank$

Let's look at this through an example.

The Complet Solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2 & \chi_1 + 4\chi_0 + 7 \\ -4\chi_0 - 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} + \chi_1 \begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix}$

 $= -2x_1 + 4x_4 = 7$ $= 4 - (-3 - 1x_4) - 2x_1$

X2. X4 are free variables!

X1. X3 are pivot Variable!

special solution (put. X2= X4=0) Null space!

Example

Solve
$$\begin{bmatrix}
1 & 2 & 1 & 0 \\
2 & 4 & 4 & 8 \\
4 & 8 & 6 & 8
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}.
\begin{bmatrix}
1 & 2 & 1 & 0 \\
2 & 4 & 4 & 8 \\
4 & 8 & 6 & 8
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
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\begin{bmatrix}
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4 & 8 & 6 & 8
\end{bmatrix}
\begin{bmatrix}
4 \\ 2 \\ 10 \end{bmatrix}.
\begin{bmatrix}
1 & 2 & 1 & 0 \\
4 & 8 & 6 & 8
\end{bmatrix}
\begin{bmatrix}
2 & 1 & 2 & 1 & 0 \\
2 & 3 & -42 & 1
\end{bmatrix}
= -2x_1 + 4x_4 = 7$$

$$= (4 - (-3 - 1)x_0) - 2$$

$$= (4$$

Example

Solve
$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}.$$
 4 Unichle, 3 Equations
$$A \in \mathbb{R}^{4 \times 3}.$$
 What is the largest possible rank? 3
$$A \in \mathbb{R}^{2 \times 4},$$
 What is the largest possible rank? 3
$$A \in \mathbb{R}^{3 \times 4},$$
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$$A \in \mathbb{R}^{3 \times 4},$$
 What is the largest possible rank? 3

Particular vs. Nullspace Solution

- The particular solution is obtained by setting the free variables to zero. The particular solution solves $A\vec{x}_p = \vec{b}$.
- The nullspace solution is obtained by setting the right-hand-side, i.e., the vector \vec{b} , to $\vec{0}$. There are n-r nullspace solutions which solve $A\vec{x}_n = \vec{0}$.
- The complete solution to $A\vec{x} = \vec{b}$ can be expressed as $\vec{x} = \vec{x}_p + \vec{x}_n$.



More on Ranks and Systems

Rank

Recall: the rank r of a matrix A is the number of pivots.

- The r pivot columns are linearly independent.
- There are n-r free columns.
- Since $Col A=span\{pivot columns\}$, then the column space spans an r-dimensional space.
- The "dimension" of the nullspace is n-r.

Full Column Rank r = n

Suppose A is an $m \times n$ matrix. Then A has full column rank r = n if:

- all columns of A are pivot columns
- all columns of A are linearly independent
- there are no free columns \implies no free solutions
- Nul $A = \{\vec{0}\}$ # Free Uniables = n r = 0
- if $A\vec{x} = \vec{b}$ has a solution, then it has exactly one solution.

Full Row Rank r = m

Suppose A is an $m \times n$ matrix. Then A has full row rank r = m if:

- all rows of A are linearly independent
- there are n-r=n-m nullspace solutions

• there are
$$n-r=n-m$$
 nullspace solutions

• The column space of A spans all of
$$\mathbb{R}^m$$

$$A\vec{x} = \vec{b} \text{ has a solution for every } \vec{b} \qquad \begin{pmatrix} \vec{b}_1 \\ \vec{b}_3 \end{pmatrix} = C \begin{pmatrix} \vec{o} \\ \vec{o} \end{pmatrix} + C_1 \begin{pmatrix} * \\ * \end{pmatrix} + C_3 \begin{pmatrix} * \\ * \end{pmatrix} +$$

Col (A) = 1R m

span { pivot Glavny }

```
A x = 6
    A E IR mx n
  (1) m < n => { 0 Solution }
              Pank < m < n # Free Uniables = n - Pan +>0
  \bigcirc m = \eta A is a square matrix
                                                   A is full Glumn lank
       \perp. tank = m = n
                1, # Free = n - 12m k = 0 => Nul (A) = { }
               2. A is also full now rank (rank=m)
                         (A) = RM
                            and Ax = 6 always have a Solution
         A \times = b have and only have single solution.
                  & A has a inverse Metrix!!!
                - now are linear indepedent 

Tow fall rank
                - Cols are linear independent
                                                    e 6 full rank
                                                      every a ica pivot a
      \frac{2}{n} rank < m = n
                  # Free = N-CAN + >0 \Rightarrow SO Solutions
    # f Wonable
(3) m > n
               largest possible rank = N
                     # Free Utriable : n-rank =0
                                                              6 BCACA
                                                            Solutions

1 to begin (A)

Solutions
           2. rank < n < m
                    # Free Variables = n- Rank >0
```

Whether Ax = b have Solution Should use be sol (A)

(1)
$$r = m, r = n \implies \text{square invertible} \implies A\vec{x} = \vec{b} \text{ has 1 solution}$$

(2)
$$r = m, r < n \implies$$
 short and wide $\implies A\vec{x} = \vec{b}$ has an infinite number of solutions

(3)
$$r < m, r = n \implies \text{tall and thin} \implies A\vec{x} = \vec{b} \text{ has } 0 \text{ or } 1 \text{ solution}$$

(4)
$$r < m, r < n \implies$$
 not full rank $\implies A\vec{x} = \vec{b}$ has either an infinite number of solutions or no solutions

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