

Linear Algebra

Lecture 9 Complete Solutions

Dr. Ralph Chikhany



Strang Sections 3.3 – The Complete Solutions to Ax = B

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

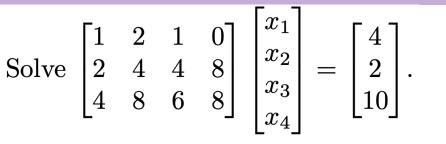


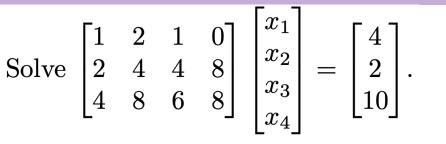
Complete Solutions

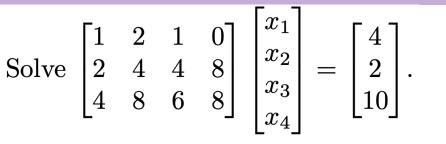
Theorem

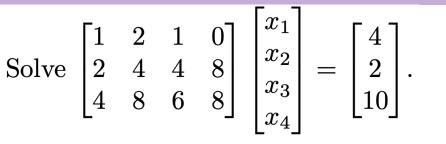
Let A be an $m \times n$ matrix such that m < n. In this case, we are guaranteed to have free columns, and the system $A\vec{x} = \vec{b}$ will have more unknowns than equations, so it will have free variables associated with the free columns. Thus, this system will always have either an *infinite number* of solutions or *no* solutions.

Let's look at this through an example.









Particular vs. Nullspace Solution

- The particular solution is obtained by setting the free variables to zero. The particular solution solves $A\vec{x}_p = \vec{b}$.
- The nullspace solution is obtained by setting the right-hand-side, i.e., the vector \vec{b} , to $\vec{0}$. There are n r nullspace solutions which solve $A\vec{x}_n = \vec{0}$.

• The complete solution to $A\vec{x} = \vec{b}$ can be expressed as $\vec{x} = \vec{x}_p + \vec{x}_n$.



More on Ranks and Systems

Rank

Recall: the rank r of a matrix A is the number of pivots.

- The r pivot columns are linearly independent.
- There are n r free columns.
- Since Col A=span{pivot columns}, then the column space spans an r-dimensional space.
- The "dimension" of the nullspace is n-r.

Full Column Rank r = n

Suppose A is an $m \times n$ matrix. Then A has full column rank r = n if:

- all columns of A are pivot columns
- all columns of A are linearly independent
- there are no free columns \implies no free solutions
- Nul $A = {\vec{0}}$
- if $A\vec{x} = \vec{b}$ has a solution, then it has exactly one solution.

Full Row Rank r = m

Suppose A is an $m \times n$ matrix. Then A has full row rank r = m if:

- all rows of A have pivot positions
- all rows of A are linearly independent
- there are n r = n m nullspace solutions
- The column space of A spans all of \mathbb{R}^m
- $A\vec{x} = \vec{b}$ has a solution for every \vec{b}

Let A be an $m \times n$ matrix with rank r. The solutions to $A\vec{x} = \vec{b}$ can be classified as follows:

(1)
$$r = m, r = n \implies$$
 square invertible $\implies A\vec{x} = \vec{b}$ has 1 solution

(2) $r = m, r < n \implies$ short and wide $\implies A\vec{x} = \vec{b}$ has an infinite number of solutions

(3) $r < m, r = n \implies$ tall and thin $\implies A\vec{x} = \vec{b}$ has 0 or 1 solution

(4) $r < m, r < n \implies$ not full rank $\implies A\vec{x} = \vec{b}$ has either an infinite number of solutions or no solutions

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