Lecture 9

Complete Solutions

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Strang Sections 3.3 – The Complete Solutions to $Ax = B$

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Complete Solutions

Theorem

Let A be an $m \times n$ matrix such that $m < n$. In this case, we are guaranteed to have free columns, and the system $A\vec{x} = \vec{b}$ will have more unknowns than equations, so it will have free variables associated with the free columns. Thus, this system will always have either an *infinite number* of solutions or *no* solutions.

Let's look at this through an example.

Example

Solve $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 4 & 8 \\ 4 & 8 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 10 \end{bmatrix}$.

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Particular vs. Nullspace Solution

- The particular solution is obtained by setting the free variables to zero. The particular solution solves $A\vec{x}_p = \vec{b}$.

- The nullspace solution is obtained by setting the right-hand-side, i.e., the vector $\vec{b}$, to $\vec{0}$. There are $n - r$ nullspace solutions which solve $A\vec{x}_n = \vec{0}$.

- The complete solution to $A\vec{x} = \vec{b}$ can be expressed as $\vec{x} = \vec{x}_p + \vec{x}_n$.

More on Ranks and Systems

Rank

Recall: the rank r of a matrix A is the number of pivots.

- The r pivot columns are linearly independent.
- There are $n - r$ free columns.
- Since Col A=span{pivot columns}, then the column space spans an r-dimensional space.
- The "dimension" of the nullspace is $n - r$.

Full Column Rank $r = n$

Suppose A is an $m \times n$ matrix. Then A has full column rank $r = n$ if:

- all columns of A are pivot columns
- all columns of A are linearly independent
- there are no free columns $\implies$ no free solutions
- $\operatorname{Nul} A = \{\vec{0}\}$
- if $A\vec{x} = \vec{b}$ has a solution, then it has exactly one solution.

Full Row Rank $r = m$

Suppose A is an $m \times n$ matrix. Then A has full row rank $r = m$ if:

- all rows of A have pivot positions
- all rows of A are linearly independent
- there are $n - r = n - m$ nullspace solutions
- The column space of A spans all of $\mathbb{R}^m$
- $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$

Rank and Solvability

Let A be an $m \times n$ matrix with rank r. The solutions to $A\vec{x} = \vec{b}$ can be classified as follows:

(1) $r = m,\ r = n \implies$ square invertible $\implies A\vec{x} = \vec{b}$ has 1 solution

(2) $r = m,\ r < n \implies$ short and wide $\implies A\vec{x} = \vec{b}$ has an infinite number of solutions

(3) $r < m,\ r = n \implies$ tall and thin $\implies A\vec{x} = \vec{b}$ has 0 or 1 solution

(4) $r < m,\ r < n \implies$ not full rank $\implies A\vec{x} = \vec{b}$ has either an infinite number of solutions or no solutions

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