Lecture 8
Nullspace

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Strang Sections 3.2 – The nullspace of A

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

The Nullspace of a Matrix

Nullspace

Let A be an $m \times n$ matrix, such that $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n]$, where $\vec{a}_i \in \mathbb{R}^m$ $(1 \leq i \leq n)$. The nullspace of A consists of all solutions $\vec{x}$ to $A\vec{x} = \vec{0}$. That is,

$$\operatorname{Nul} A = \left\{ \vec{x} \,\middle|\, A\vec{x} = \vec{0} \right\}.$$

Compute the nullspace of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 2 \\ 1 & 3 & 1 & 2 \end{bmatrix}$.

Theorem

The nullspace of A is a subspace of $\mathbb{R}^n$.

Row Echelon Form (REF)

REF

An $m \times n$ matrix is in echelon or row-echelon form (REF) if:

(1) All rows consisting entirely of zeros lie beneath all nonzero rows.

(2) The first nonzero element in any row is called a **pivot**.

(3) Any pivot must lie to the right of any pivot above it.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot Columns vs. Free Columns

Pivot Columns: The first pivot column in any matrix A is the left-most nonzero column. The top element in that column is called the first *pivot* or *pivot position*. If the first pivot is zero, we swap rows to get a nonzero pivot. A pivot column in a matrix in REF is a column that contains *exactly* one pivot.

Free Columns: A free column in a matrix in REF is a column that contains *no* pivots.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 7 \\ 2 & 5 & 9 \end{bmatrix}$. Write A in row-echelon form and find its nullspace.

Reduced Row Echelon Form (RREF)

RREF

To find the reduced row-echelon form (RREF) of a matrix A:

(1) First find the REF of A. That is, find the *pivots* and use them to make all elements *below* them equal zero.

(2) Then, use the obtained *pivots* to make all elements *above* them equal zero.

(3) Lastly, make all pivots equal 1.

This procedure does not change the nullspace of the original matrix A.

Example

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 7 \\ 2 & 5 & 9 \end{bmatrix}$. Write A in reduced row-echelon form and find its nullspace.

Rank

What is the Rank of a Matrix?

The rank of an $m \times n$ matrix A is the number of pivots. We call this number r.

We think of the rank as the "true size" of a matrix. For example, the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ is 2×3, however, its REF is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. That is, the second row is not giving any additional information; it is just a copy of the first row.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 7 \\ 2 & 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 1 \\ 0 & 1 & 2 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$