Linear Algebra

Lecture 7

Vector Spaces and Subspaces

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Quiz 3

1. M. N are Matrix. $MN = N$. Then $M = I$ ✗

1×1 matrix (real number) $mn = n$ then $m = 1$ ✗

there is also possiblity $n = 0$

If N has an inverse matrix, then $MN = N$

$MNN^{-1} = NN^{-1} \Rightarrow M = I$

What if N don't have an inverse?

$MN - N = 0$ $(M - I)N = 0 \not\Rightarrow M - I = 0$

$MN - I \cdot N = (M - I) \cdot N$

Today: What is the set of all possible $|M - I|$ Vector Space!

Set $\{x \mid Nx = 0,\ x \in \mathbb{R}^d\}$ is Vector Space

the set of all possible x that satisfies $Nx = 0,\ x \in \mathbb{R}^d$

(such as)

Strang Sections 3.1 – Spaces of Vectors

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Vector Spaces

Vector Spaces

$\mathbb{R}$ is closed r.p.t. $+$ $a \in \mathbb{R}$, $b \in \mathbb{R}$, the $a+b \in \mathbb{R}$

holds for all c_1, c_2 $\mathbb{R}$

vector space V means it is closed r.p.t. linear Combination: if means $v_1, v_2 \in V$, then $c_1 v_1 + c_2 v_2 \in V$

A vector space V defined over a field $\mathbb{F}$ ($\mathbb{R}$ in our case) consists of a set on which addition and scalar multiplication are defined so that for each pair of elements v and w in V, there is a unique element $v + w \in V$, and for each element $c \in \mathbb{R}$ and $v \in V$, there is a unique element $cv \in V$, s.t. the following conditions hold:

1. addition 2. scalar multiplication.

(VS1) For all $v, w \in V$, $v + w = w + v$.

(VS2) For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$.

$0 \in V$

(VS3) There exists an element in V denoted by 0, s.t. $v + 0 = v$ for each $v \in V$.

"negative"

(VS4) For each element $v \in V$, there exists an element $w \in V$, s.t. $v + w = 0$.

if 1. 2. Statisfies VS1 – VS8 then we call it a vector space.

defined for addition

Vector Spaces

A vector space V defined over a field $\mathbb{F}$ ($\mathbb{R}$ in our case) consists of a set on which addition and scalar multiplication are defined so that for each pair of elements v and w in V, there is a unique element $v + w \in V$, and for each element $c \in \mathbb{R}$ and $v \in V$, there is a unique element $cv \in V$, s.t. the following conditions hold:

closed!

(VS5) For each element $v \in V$, $1v = v$.

(VS6) For each pair of elements $c, d \in \mathbb{R}$, and each $v \in V$, $(cd)v = c(dv)$.

(VS7) For each element $c \in \mathbb{R}$, and each pair $v, w \in V$, $c(v + w) = cv + cw$.

(VS8) For each pair of elements $c, d \in \mathbb{R}$, and each $v \in V$, $(c + d)v = cv + dv$.

$\Downarrow$

defined for linear combination

Vector Spaces

(VS1) For all $v, w \in V$, $v + w = w + v$.

(VS2) For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$.

(VS3) There exists an element in V denoted by 0, s.t. $v + 0 = v$ for each $v \in V$.

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(VS7) For each element $c \in \mathbb{R}$, and each pair $v, w \in V$, $c(v + w) = cv + cw$.

(VS8) For each pair of elements $c, d \in \mathbb{R}$, and each $v \in V$, $(c + d)v = cv + dv$.

Note: All elements in the field $\mathbb{R}$ are called scalars and all elements in the vector space V are called vectors.

Example

Let S be a non-empty set, and let $\mathcal{F}(S, \mathbb{R})$ denote the set of all functions from S to $\mathbb{R}$. Two functions $f, g \in \mathcal{F}$ are called equal if $f(s) = g(s)$ for all $s \in S$. Show that the set $\mathcal{F}(S, \mathbb{R})$ is a vector space with the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$ by

$$(f+g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c[f(s)]$$

for all $s \in S$.

Example.
- $\mathbb{R}^d$ is a vector space!
- all the functions. $f: S \to \mathbb{R}$ (set) is a vector space.

$S = \{1, 2, 3\}$

understand a function as a vector. $f: S \to \mathbb{R}$. understand f as $\begin{bmatrix} f(1) \\ f(2) \\ f(3) \end{bmatrix}$

1. What is $f+g$

$f \leftarrow \begin{bmatrix} f(1) \\ f(2) \\ f(3) \end{bmatrix} \vec{v_1}$ $\quad g \leftarrow \begin{bmatrix} g(1) \\ g(2) \\ g(3) \end{bmatrix} \vec{v_2}$

$\underset{\vec{v_1} + \vec{v_2}}{\begin{bmatrix} f(1) + g(1) \\ f(2) + g(2) \\ f(3) + g(3) \end{bmatrix}} \to = \begin{bmatrix} (f+g)(1) \\ (f+g)(2) \\ (f+g)(3) \end{bmatrix} \leftarrow (f+g)$

2. What is $c \in \mathbb{R}$, $c \cdot f$

$\downarrow$ Similar

$(cf)(s) = c \cdot [f(s)]$

Verify <u>VS1 – VS8</u>

VS1. $f+g = g+f$

check $(f+g)(s) = (g+f)(s)$ holds for every s

$f(s) + g(s) = g(s) + f(s)$

Example

$\in$: belongs to

$\forall$: for all

$\exists$: there exists

WTS : want To Show

Let S be a non-empty set, and let $\mathcal{F}(S,\mathbb{R})$ denote the set of all functions from S to $\mathbb{R}$. Two functions $f, g \in \mathcal{F}$ are called equal if $f(s) = g(s)$ for all $s \in S$. Show that the set $\mathcal{F}(S,\mathbb{R})$ is a vector space with the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$ by

Verify VS1-8
Not Required

$$(f+g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c[f(s)]$$

for all $s \in S$. Let $F(S,\mathbb{R}) = F$.

(VS1) $\underline{\forall v, w \in V,\ v+w = w+v}$ WTS $\forall f, g \in F$, $f+g = g+f$.

First, observe that $(f+g)(s) = f(s)+g(s)$ and $(g+f)(s) = g(s)+f(s)$.

But $f(s)+g(s) = g(s)+f(s)$ because $f(s) \in \mathbb{R}$ and $g(s) \in \mathbb{R}$, and addition is commutative on the real numbers.

(VS2) $\underline{\forall u, v, w \in V,\ (u+v)+w = u+(v+w)}$. Let $f, g, h \in F$.

Then $((f+g)+h)(s) = \underbrace{(f+g)(s)} + h(s) = \underbrace{f(s)+g(s)} + h(s)$

$(f+(g+h))(s) = f(s) + \underbrace{(g+h)(s)} = f(s) + \underbrace{g(s)+h(s)}$

Equal because $f(s), g(s), h(s) \in \mathbb{R}$

Example

Let S be a non-empty set, and let $\mathcal{F}(S, \mathbb{R})$ denote the set of all functions from S to $\mathbb{R}$. Two functions $f, g \in \mathcal{F}$ are called equal if $f(s) = g(s)$ for all $s \in S$. Show that the set $\mathcal{F}(S, \mathbb{R})$ is a vector space with the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$ by

$$(f+g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c[f(s)]$$

for all $s \in S$.

Verify VS1-8
Not Required

(VS3) $\exists\, 0 \in V$ s.t. $0 + v = v \quad \forall v \in V$

Let $f \in F$. Define $\underbrace{0(s)}_{\text{function}} = \underbrace{0}_{\text{real number}} \quad \forall s \in F$. Then

the zero function, any function f

$(0 + f)(s) = 0(s) + f(s)$

any input $s \in S$

$= \underbrace{0}_{\in \mathbb{R}} + \underbrace{f(s)}_{\in \mathbb{R}}$

$(0+f)(s) = f(s)$

functions

(VS4) $\forall v \in V,\ \exists\, w \in V$ s.t. $\overbrace{v + w = 0}^{\text{functions}}$

Let $f \in F$. Then $cf \in F$ for any $c \in \mathbb{R}$. Let $c = -1$ and define $g = -f$ such that $g(s) = -f(s) \quad \forall s \in S$. Then $(f+g)(s) = f(s) + g(s)$

$= f(s) + (-f(s)) = \underbrace{0}_{\in \mathbb{R}}$

Thus $f + g = 0 \rightarrow$ function

Example

Let S be a non-empty set, and let $\mathcal{F}(S,\mathbb{R})$ denote the set of all functions from S to $\mathbb{R}$. Two functions $f, g \in \mathcal{F}$ are called equal if $f(s) = g(s)$ for all $s \in S$. Show that the set $\mathcal{F}(S,\mathbb{R})$ is a vector space with the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$ by

Verify VS1-8
Not Required

$$(f+g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c[f(s)]$$

for all $s \in S$.

$\in \mathbb{R}$

(VS5) $\underline{\forall v \in V,\ 1 \cdot v = v}$

Let $f \in F$ and $c = 1$. Then $(cf)(s) = c[f(s)] = \underset{\in \mathbb{R}}{1} \cdot \underset{\in \mathbb{R}}{f(s)} = f(s)$

(VS6) $\underline{\forall c,d \in \mathbb{R},\ \forall v \in V,\ (cd) \cdot v = c \cdot (dv)}$

Let $f \in F$, $c, d \in \mathbb{R}$. Then $\forall s \in S$:

$\in \mathbb{R}$ $\in \mathbb{R}$ $\in \mathbb{R}$

$[(cd)f](s) = (cd)f(s) = c \cdot d \cdot f(s)$

$\in \mathbb{R}$ $\in \mathbb{R}$ $\in \mathbb{R}$

$[c(df)](s) = c(df(s)) = c \cdot (d \cdot f(s)) = c \cdot d \cdot f(s)$

Same output

Associative prop. of scalar multiplication

Example

Let S be a non-empty set, and let $\mathcal{F}(S, \mathbb{R})$ denote the set of all functions from S to $\mathbb{R}$. Two functions $f, g \in \mathcal{F}$ are called equal if $f(s) = g(s)$ for all $s \in S$. Show that the set $\mathcal{F}(S, \mathbb{R})$ is a vector space with the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$ by

Verify VS1-8
Not Required

$$(f+g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c[f(s)]$$

for all $s \in S$.

(VS7) $\forall c \in \mathbb{R}, \forall v, w \in V, \ c(v+w) = cv + cw$

Let $f, g \in F$ and $c \in \mathbb{R}$. Then

$$[c(f+g)](s) \underset{\text{by definition}}{=} c[(f+g)(s)] \underset{\text{by definition}}{=} c[\underset{\in\mathbb{R}}{f(s)} + \underset{\in\mathbb{R}}{g(s)}] = cf(s) + cg(s)$$

by distributive prop. of real numbers (scalar) multiplication over scalar addition

(VS8) $\forall c, d \in \mathbb{R}, \forall v \in V, \ (c+d)v = cv + dv$

Let $f \in F$ and $c, d \in \mathbb{R}$. Then

$$[(c+d)f](s) = (\underset{\in\mathbb{R}}{c} + \underset{\in\mathbb{R}}{d})(\underset{\in\mathbb{R}}{f(s)}) = cf(s) + df(s)$$

(by distributive prop. of real numbers (scalar) multip. over scalar addition)

Example

Let S be a non-empty set, and let $\mathcal{F}(S, \mathbb{R})$ denote the set of all functions from S to $\mathbb{R}$. Two functions $f, g \in \mathcal{F}$ are called equal if $f(s) = g(s)$ for all $s \in S$. Show that the set $\mathcal{F}(S, \mathbb{R})$ is a vector space with the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$ by

$$(f+g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c[f(s)]$$

for all $s \in S$.

check using

vector space means it is closed w.r.t. linear combination.

Example

1 $\{x \mid Ax = 0\}$ is a vector space ✓

2 $\{x \mid Ax = b\}$ $(b \neq 0)$ is a vector space ✗

1. If $v_1, v_2 \in \{x \mid Ax = 0\}$ ← this means $Av_1 = \vec{0}$, $Av_2 = \vec{0}$

check. $c_1v_1 + c_2v_2 \in \{x \mid Ax = 0\}$ ← this means we need to check $A(c_1v_1 + c_2v_2) = \vec{0}$

$A(c_1\vec{v_1} + c_2\vec{v_2}) = c_1 \underbrace{A\vec{v_1}}_{\vec{0}} + c_2 \underbrace{A\vec{v_2}}_{\vec{0}} = c_1\vec{0} + c_2\vec{0} = \vec{0}$ ✓

2. $v_1, v_2 \in \{x \mid Ax = b\}$ ← $A\vec{v_1} = \vec{b}$ $\quad A\vec{v_2} = \vec{b}$ $\quad$ $(c_1 + c_2)\vec{b}$ is not always $\vec{b}$

? $(c_1v_1 + c_2v_2) \in \{x \mid Ax = b\}$ ← check. $A(c_1v_1 + c_2v_2) = \vec{b}$ ✗

Null space. Vector space means it is closed w.r.t. linear Combination.

Example

1. $\{x \mid Ax = 0\}$ is a vector space ✓

2. $\{x \mid Ax = b\}$ $(b \neq 0)$ is a vector space ✗

3. $\text{span}\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$ is a vector space ✓

$v_1, v_2 \in \text{span}\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$

This mean $\vec{v}_1 = c_{11}\vec{x}_1 + c_{12}\vec{x}_2 + \cdots + c_{1n}\vec{x}_n$

$\vec{v}_2 = c_{21}\vec{x}_1 + c_{22}\vec{x}_2 + \cdots + c_{2n}\vec{x}_n$

Check. $c_1v_1 + c_2v_2 \in \text{span}\{\vec{x}_1, \vec{x}_2, \cdots, \vec{x}_n\}$

Whether $c_1\vec{v}_1 + c_2\vec{v}_2$ is L.C. of $\vec{x}_1 \cdots \vec{x}_n$

$$c_1\vec{v}_1 + c_2\vec{v}_2 = c_1\left[c_{11}\vec{x}_1 + c_{12}\vec{x}_2 + \cdots + c_{1n}\vec{x}_n\right] + c_2\left[c_{21}\vec{x}_1 + c_{22}\vec{x}_2 + \cdots + c_{2n}\vec{x}_n\right]$$

$$= \underbrace{(c_1c_{11} + c_2c_{21})\vec{x}_1 + (c_1c_{12} + c_2c_{22})\vec{x}_2 + \cdots + (c_1c_{1n} + c_2c_{2n})\vec{x}_n}_{\text{is a linear Combination of } \vec{x}_1 \cdots \vec{x}_n}$$

Example

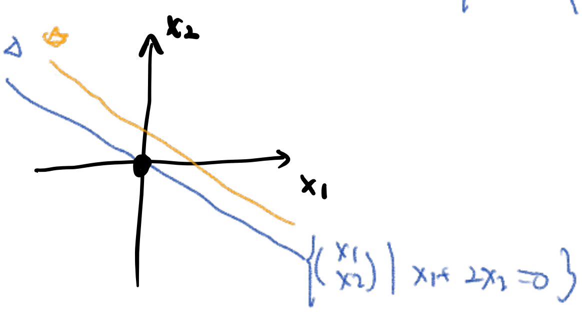
Vector Space means a line / plane / higher dimensional generalization ... that go through the origin

- $A \in \mathbb{R}^{1\times 2}$ $A = [1 \quad 2]$

the Null space. $\{\vec{x} \in \mathbb{R}^2 \mid Ax = \vec{0}\}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ Then $Ax = x_1 + 2x_2$

means $\left\{\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 + 2x_2 = 0\right\}$

$x_2 = -\frac{1}{2}x_1$



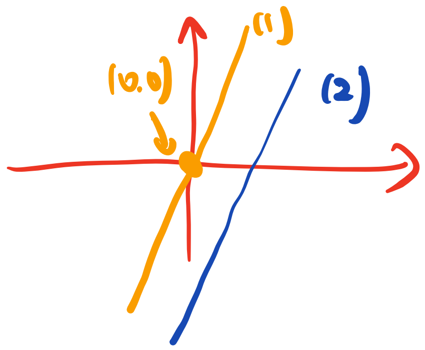
$\{\vec{x} \in \mathbb{R}^2 \mid Ax = 1\}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$Ax_1 = x_1 + 2x_2 = 1$

Ex. $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $A \cdot v = v_1 + 2v_2$

1) $\left\{ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \middle| v_1 + 2v_2 = 0 \right\}$

2) $\left\{ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \middle| v_1 + 2v_2 = b \right\}$ $b \neq 0$



Vector space means "line/plane" go through the origin !!

[by definition. $\vec{0} \in V$)

VS 3 !

$v_1 \cdots v_n \in V$

$\Rightarrow c_1 v_1 + \cdots + c_n v_n \in V$

Null space !

!!! $\{ x \mid Ax = 0 \}$ is a vector space

$\{ x \mid Ax = b \}$, $b \neq 0$ is not a Vector space

$\{ Ax \mid x \in \mathbb{R}^n \}$ is a vector space

$\| $

$Col(A) = span\{ \vec{a}_1 \cdots \vec{a}_n \}$, $A = [\vec{a}_1 \cdots \vec{a}_n]$

Subspaces

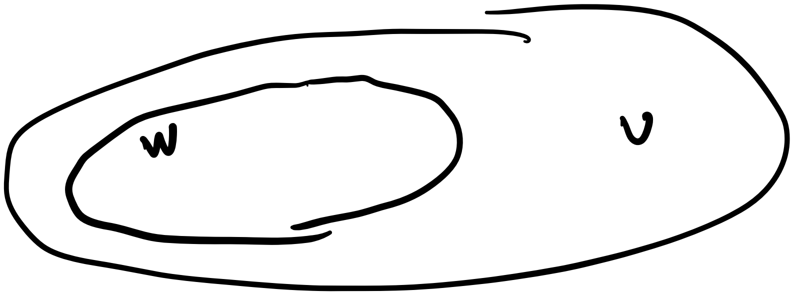
Solving Systems of Equations

A set $W \subset V$ is a subspace of a vector space V if for all vectors $v, w \in W$ and $c \in \mathbb{R}$ if

(1) $v + w \in W$

(2) $cv \in W$

(3) $\mathbf{0} \in W$



W is a vector space

V is another vector space

$W \subseteq V$ (use the same addition and scalar multiplication rule)

W itself is a vector space.

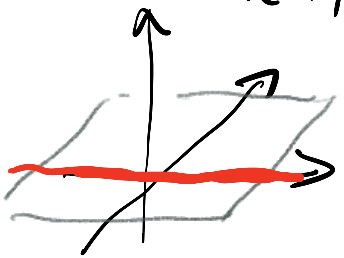
W is the subspace of V

Ex. $\mathbb{R}^3$ is a vector space

The xy-plane a vector space

xy-plane is a subspace of $\mathbb{R}^3$

$\left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\}$ ← check by yourself



x-axis is a vector space

$\left\{ \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \middle| \, x \in \mathbb{R} \right\}$ ← is a vector space

x-axis is a subspace of $\mathbb{R}^3$ / xy-plane.

Example

Consider the vector space $\mathbb{M}_{2\times 2}(\mathbb{R})$. Show that U (the set of all upper triangular matrices) and D (the set of all diagonal matrices) are subspaces of $\mathbb{M}_{2\times 2}(\mathbb{R})$.

2x2 Matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

2x2 diag Matrix $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ is the subspace of $\mathbb{M}_{2\times 2}$

$$c_1\begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix} + c_2\begin{pmatrix} a_2 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} c_1a_1 + c_2a_2 & \\ & c_1d_1 + c_2d_2 \end{pmatrix}$$

Example

Consider the vector space $\mathbb{M}_{2\times 2}(\mathbb{R})$. Show that U (the set of all upper triangular matrices) and D (the set of all diagonal matrices) are subspaces of $\mathbb{M}_{2\times 2}(\mathbb{R})$.

Column Space

Column Space

all column vectors

Let $A \in \mathbb{M}_{m\times n}(\mathbb{R})$, such that $A = [\vec{a}_1 \quad \vec{a}_2 \quad \dots \quad \vec{a}_n]$, where $\vec{a}_i \in \mathbb{R}^m$ $(1 \le i \le n)$. The column space of A consists of all possible linear combinations of the columns of A. That is,

$$\text{Col}\, A = \text{span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$$

is a vector space

span of column vectors

To solve $A\vec{x} = \vec{b}$, you must express $\vec{b}$ as a linear combination of the columns of A. Thus, $\vec{b}$ has to be in the column space of A, otherwise we won't be able to find a solution for the system $A\vec{x} = \vec{b}$.

$A\vec{x} = \vec{b}$ have solution. $\Leftrightarrow$ $\vec{b} \in \text{Col}(A)$

This means: $\{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\} = \text{Col}\, A = \text{span}\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n\}$ is a vector space

Column Space

The column space of A is a subspace of $\mathbb{R}^m$.

for linear system $Ax = b$ have solution $\Leftrightarrow$ b lies in the column space of A

$b \in \text{Col}(A)$

$$A = [\vec{a}_1 \quad \vec{a}_2 \quad \dots \quad \vec{a}_n] = \left[\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right]$$

$$\text{Therefore, Col } A = \text{span} \left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\} \subset \mathbb{R}^m.$$

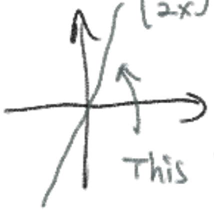
Example – Describe the Column Space of A

Example. What is the Col(A)

1. $A = \begin{bmatrix} 1 & -3 \\ 2 & -6 \end{bmatrix}$ $\left[\begin{array}{cc|c} 1 & -3 & x \\ 2 & -6 & 2x \end{array}\right]$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} -3 \\ -6 \end{pmatrix}$ $\vec{v}_2 = -3\vec{v}_1$

$\text{span}\{\vec{v}_1, \vec{v}_2\} = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$



$= \left\{\begin{pmatrix} x \\ 2x \end{pmatrix} \middle| x \in \mathbb{R}\right\}$

This is the Column Space

2. $A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ pivots

"extra Vector"

" Upper Triangular Form "

$\text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix}\right\} = \mathbb{R}^3$

why ??

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} \begin{matrix} \leftarrow c_1 \\ \leftarrow c_2 \\ \leftarrow c_3 \end{matrix}$

$\text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}\right\} = \mathbb{R}^3$

3. $A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}\right\}$

$\text{Col}(A) := \left\{\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \middle| x_1, x_2 \in \mathbb{R}\right\}$ can already given you

Example – Describe the Column Space of A