Lecture 7

Vector Spaces and Subspaces

Dr. Ralph Chikhany

Strang Sections 3.1 – Spaces of Vectors

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

Vector Spaces

Vector Spaces

A vector space V defined over a field $\mathbb{F}$ ($\mathbb{R}$ in our case) consists of a set on which addition and scalar multiplication are defined so that for each pair of elements v and w in V, there is a unique element $v + w \in V$, and for each element $c \in \mathbb{R}$ and $v \in V$, there is a unique element $cv \in V$, s.t. the following conditions hold:

(VS1) For all $v, w \in V$, $v + w = w + v$.

(VS2) For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$.

(VS3) There exists an element in V denoted by 0, s.t. $v + 0 = v$ for each $v \in V$.

(VS4) For each element $v \in V$, there exists an element $w \in V$, s.t. $v + w = 0$.

Vector Spaces

A vector space V defined over a field $\mathbb{F}$ ($\mathbb{R}$ in our case) consists of a set on which addition and scalar multiplication are defined so that for each pair of elements v and w in V, there is a unique element $v + w \in V$, and for each element $c \in \mathbb{R}$ and $v \in V$, there is a unique element $cv \in V$, s.t. the following conditions hold:

(VS5) For each element $v \in V$, $1v = v$.

(VS6) For each pair of elements $c, d \in \mathbb{R}$, and each $v \in V$, $(cd)v = c(dv)$.

(VS7) For each element $c \in \mathbb{R}$, and each pair $v, w \in V$, $c(v + w) = cv + cw$.

(VS8) For each pair of elements $c, d \in \mathbb{R}$, and each $v \in V$, $(c + d)v = cv + dv$.

Vector Spaces

(VS1) For all $v, w \in V$, $v + w = w + v$.

(VS2) For all $u, v, w \in V$, $(u + v) + w = u + (v + w)$.

(VS3) There exists an element in V denoted by 0, s.t. $v + 0 = v$ for each $v \in V$.

(VS4) For each element $v \in V$, there exists an element $w \in V$, s.t. $v + w = 0$.

(VS5) For each element $v \in V$, $1v = v$.

(VS6) For each pair of elements $c, d \in \mathbb{R}$, and each $v \in V$, $(cd)v = c(dv)$.

(VS7) For each element $c \in \mathbb{R}$, and each pair $v, w \in V$, $c(v + w) = cv + cw$.

(VS8) For each pair of elements $c, d \in \mathbb{R}$, and each $v \in V$, $(c + d)v = cv + dv$.

Note: All elements in the field $\mathbb{R}$ are called scalars and all elements in the vector space V are called vectors.

Example

Let S be a non-empty set, and let $\mathcal{F}(S, \mathbb{R})$ denote the set of all functions from S to $\mathbb{R}$. Two functions $f, g \in \mathcal{F}$ are called equal if $f(s) = g(s)$ for all $s \in S$. Show that the set $\mathcal{F}(S, \mathbb{R})$ is a vector space with the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$ by

$$(f+g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c[f(s)]$$

for all $s \in S$.

Example

Let S be a non-empty set, and let $\mathcal{F}(S, \mathbb{R})$ denote the set of all functions from S to $\mathbb{R}$. Two functions $f, g \in \mathcal{F}$ are called equal if $f(s) = g(s)$ for all $s \in S$. Show that the set $\mathcal{F}(S, \mathbb{R})$ is a vector space with the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}$ and $c \in \mathbb{R}$ by

$$(f+g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c[f(s)]$$

for all $s \in S$.

Subspaces

Solving Systems of Equations

A set $W \subset V$ is a subspace of a vector space V if for all vectors $v, w \in W$ and $c \in \mathbb{R}$ if

(1) $v + w \in W$

(2) $cv \in W$

(3) $\mathbf{0} \in W$

W itself is a vector space.

Example

Consider the vector space $\mathbb{M}_{2\times 2}(\mathbb{R})$. Show that U (the set of all upper triangular matrices) and D (the set of all diagonal matrices) are subspaces of $\mathbb{M}_{2\times 2}(\mathbb{R})$.

Example

Consider the vector space $\mathbb{M}_{2\times 2}(\mathbb{R})$. Show that U (the set of all upper triangular matrices) and D (the set of all diagonal matrices) are subspaces of $\mathbb{M}_{2\times 2}(\mathbb{R})$.

Column Space

Column Space

Let $A \in \mathbb{M}_{m\times n}(\mathbb{R})$, such that $A = [\vec{a}_1 \ \ \vec{a}_2 \ \ \ldots \ \ \vec{a}_n]$, where $\vec{a}_i \in \mathbb{R}^m$ $(1 \leq i \leq n)$. The column space of A consists of all possible linear combinations of the columns of A. That is,

$$\mathrm{Col}\, A = \mathrm{span}\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$$

To solve $A\vec{x} = \vec{b}$, you must express $\vec{b}$ as a linear combination of the columns of A. Thus, $\vec{b}$ has to be in the column space of A, otherwise we won't be able to find a solution for the system $A\vec{x} = \vec{b}$.

Column Space

The column space of A is a subspace of $\mathbb{R}^m$.

$$A = [\vec{a}_1 \quad \vec{a}_2 \quad \dots \quad \vec{a}_n] = \left[\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right]$$

$$\text{Therefore, Col } A = \text{span} \left\{ \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\} \subset \mathbb{R}^m.$$

Example – Describe the Column Space of A

Example – Describe the Column Space of A