Linear Algebra

Lecture 6

LU and LDU Factorizations

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Strang Sections 2.6 – Elimination = Factorization: $A = LU$ and 2.7 – Transposes and Permutations

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

LU Factorization

Example – Factorize (LU) the matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix} = L \cdot U$$

ellimination (L) · upper traingular form (U)

Step ①

$$E_{21} = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix}$$ ← Replace R2 by R2 + (−2) · R1

$$E_{21} A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 4 & 6 & 8 \end{bmatrix}$$

$= 3 + (-2) \times 1$; $3 \Leftarrow 5 + (-2) \times 1$

Step ②

$$E_{31} = \begin{bmatrix} 1 & & \\ & 1 & \\ -4 & & 1 \end{bmatrix}$$ ← Replace R3 by R3 + (−4) · R1

$$E_{31}(E_{21} A) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

Example – Factorize (LU) the matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

$$E_{31}(E_{21}A) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

Step 3)

$$E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix}$$

$$E_{32}[E_{31}E_{21}A] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \quad 4+(-2)\times 3$$

Upper Traingular Form U

Lower Traingular

$$E_{32}E_{31}E_{21}A = U$$

$$A = (E_{32}E_{31}E_{21})^{-1} \cdot U$$

$$= (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}) \cdot U$$

order?

L

Example – Factorize (LU) the matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

$$A = (E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}) \cdot U$$

order!

L

$$L = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 4 & 2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 3 \\ & & -2 \end{bmatrix}$$

diag = 1

May not be 1

Last Step Calculate L

review the inverse of elimination matrix

$$E_{21} = \begin{bmatrix} 1 & & \\ -2 & 1 & \\ & & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & & \\ & 1 & \\ -4 & & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix}$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ & & 1 \end{bmatrix} \quad E_{31}^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ 4 & & 1 \end{bmatrix} \quad E_{32}^{-1} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 2 & 1 \end{bmatrix}$$

easy

$$L = E_{21}^{-1} \cdot E_{31}^{-1} \cdot E_{32}^{-1} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 4 & 2 & 1 \end{bmatrix}$$

good order

! just need to copy non-zero element!!

just can be used for LU decomposition!!!!!

Why is this Important?

Why are LU Factorizations Important?

Consider the system $A\boldsymbol{x} = \boldsymbol{b}$ with LU factorization $A = LU$. Then we have

$$L\underbrace{U\boldsymbol{x}}_{=\boldsymbol{y}} = \boldsymbol{b}.$$

Therefore we can perform (a now familiar) 2-step solution procedure:

1. Solve the lower triangular system $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$ by forward substitution.

2. Solve the upper triangular system $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$ by back substitution.

Moreover, consider the problem $AX = B$ (i.e., many different right-hand sides that are associated with the same system matrix). In this case we need to compute the factorization $A = LU$ only once, and then

$$AX = B \quad \Longleftrightarrow \quad LUX = B,$$

and we proceed as before:

1. Solve $LY = B$ by many forward substitutions (in parallel).

2. Solve $UX = Y$ by many back substitutions (in parallel).

Solving Systems of Equations

$$\begin{aligned} 7x_1 - 2x_2 + x_3 &= 12 \\ 14x_1 - 7x_2 - 3x_3 &= 17 \\ -7x_1 + 11x_2 + 18x_3 &= 5 \end{aligned}$$

Do it at recitation

Solving Systems of Equations

Solving Systems of Equations

LDU Factorization

Goal

$$L = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 4 & 2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 1 & 1 \\ & 1 & 3 \\ & & -2 \end{bmatrix} \qquad A = L \cdot U.$$

diag = 1 (L)

May not be 1 (U)

$$A = L\, D \cdot U$$

D ↑ diag

U ↑ change U's diag to 1

$$D = \begin{pmatrix} d_{11} & & \\ & d_{22} & \\ & & d_{33} \end{pmatrix} \qquad D \cdot U = \begin{pmatrix} d_{11} \cdot U' \text{row } 1 \\ d_{22} \cdot U' \text{row } 2 \\ d_{33} \cdot U' \text{row } 3 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$$

$$A = L \cdot D \cdot U$$

U ↑ (?)

$$\begin{bmatrix} 1 & 1 & 1 \\ & 1 & 3 \\ & & -1 \end{bmatrix} \begin{matrix} \text{row } 1/1 \\ \text{row } 2/1 \\ \text{row } 3/-2. \end{matrix}$$

Let's try this with an Example

Find the LDU factorization of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

→ Find LU decomposition

→ Rewrite $U = D \cdot U'$

↑ U' diag is 1

Let's try this with an Example – Find U

Find the LDU factorization of $A = \begin{bmatrix}2 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 2\end{bmatrix}$.

LDU Decomposition

$E_{21} = \begin{bmatrix}1 & & \\ -1/2 & 1 & \\ & & 1\end{bmatrix}$

$E_{21}A = \begin{bmatrix}2 & 1 & 0\\ 0 & 3/2 & 1\\ 0 & 1 & 2\end{bmatrix}$ ($2 - 1/2 \times 1$, $1 - 1/2 \times 0$)

$E_{32} = \begin{bmatrix}1 & & \\ & 1 & \\ & -2/3 & 1\end{bmatrix}$

$E_{32}(E_{21}A) = \begin{bmatrix}2 & 1 & 0\\ 0 & 3/2 & 1\\ 0 & 0 & 4/3\end{bmatrix}$ ($2 - 2/3 \times 1$) ← This is U

$A = L \cdot U$

$L = \begin{bmatrix}1 & & \\ 1/2 & 1 & \\ & 2/3 & 1\end{bmatrix}$, $U = \begin{bmatrix}2 & 1 & 0\\ 0 & 3/2 & 1\\ & & 4/3\end{bmatrix}$ Row1/2, Row2/(3/2), Row3/(4/3) →

$D = \begin{bmatrix}2 & & \\ & 3/2 & \\ & & 4/3\end{bmatrix}$

$U' = \begin{bmatrix}1 & 1/2 & 0\\ & 1 & 2/3\\ & & 1\end{bmatrix}$

Let's try this with an Example – Find L

Find the LDU factorization of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

Symmetric Matrices

What is a Symmetric Matrix?

If $A = A^T$, then it's a symmetric Matrix

$\mathbb{R}^{n \times m}$ $\mathbb{R}^{m \times n}$

A must be a square Matrix

ex. M. symmetric

$$M = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 0 & 5 & 1 \\ 3 & 5 & -7 & -1 \\ 0 & 1 & -1 & 4 \end{bmatrix}$$

M^T: →

Properties of a Symmetric Matrix

- The inverse of a symmetric matrix is symmetric.

If $A = A^T$ then $(A^{-1})^T = A^{-1}$

pf. need to check $(A^{-1})^T A = I$ $A(A^{-1})^T = I$

$(A^{-1})^T = (A^T)^{-1} = A^{-1}$

Properties of a Symmetric Matrix

- AA^T and A^TA are symmetric.

A can by any $\mathbb{R}^{n\times m}$

$A \; A^T$ → $\mathbb{R}^{n\times n}$ square matrix

$A \in \mathbb{R}^{n\times m}$, $A^T \in \mathbb{R}^{m\times n}$

$A^T \; A$ → $\mathbb{R}^{m\times m}$

$A^T \in \mathbb{R}^{m\times n}$, $A \in \mathbb{R}^{n\times m}$

Need to check

$$(AA^T)^T = AA^T$$

$$= (A^T)^T \cdot A^T \quad \text{by } (A\cdot B)^T = B^T \cdot A^T$$

$$(A^T)^T = A$$

$$= A \cdot A^T$$

Properties of a Symmetric Matrix

- If A is symmetric, then elimination leads to $A = LDL^T$.

LDU

$$A = L \cdot D \cdot U$$

$$A^T = U^T \cdot D^T \cdot L^T$$

← another LDU Decomposition.

U^T: Lower Traingular L^T: Upper Traingular

Because LDU Decomposition is Unique!

So, $L = U^T$. $U = L^T$. → $A = L \cdot D \cdot L^T$.

Let's try this with an Example – Find D

Find the LDU factorization of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

is symmetric

$$L = \begin{bmatrix} 1 & & \\ 1/2 & 1 & \\ & 2/3 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & & \\ & 3/2 & \\ & & 4/3 \end{bmatrix} \qquad U' = \begin{bmatrix} 1 & 1/2 & 0 \\ & 1 & 2/3 \\ & & 1 \end{bmatrix}$$

!! $U' = L^T$

$A = L \cdot D \cdot U'$

$= L \cdot D \cdot L^T$ (!)

LDL decomposition!!!

Example

Factor the following symmetric matrices into $A = LDL^T$:

- $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$

$$E_{21} = \begin{bmatrix} 1 & \\ -3 & 1 \end{bmatrix} \longrightarrow L = E_{21}^{-1} = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix}$$

R2 - 3R1

$$E_{21} \cdot A = \begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix} = U$$

$$\Downarrow$$

$$D = \begin{bmatrix} 1 & \\ & -7 \end{bmatrix} \qquad U' = \begin{bmatrix} 1 & 3 \\ & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ & -7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ & 1 \end{bmatrix}$$

Example

Factor the following symmetric matrices into $A = LDL^T$:

- $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Example

Factor the following symmetric matrices into $A = LDL^T$:

- $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$PA = LU$$

PA = LU?

So far, our assumption has been that we always have nonzero pivots. In case a pivot is zero, we can't use it to eliminate elements below it, so we have to exchange rows to find a nonzero pivot before we can start eliminating.

In this case, we can't find $A = LU$. We can, however, find $PA = LU$, where P corresponds to row exchanges done on A (in advance).

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

Find PA = LU for A below

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$