NYU

Lecture 6

LU and LDU Factorizations

Dr. Yiping Lu

Strang Sections 2.6 – Elimination = Factorization: $A = LU$ and 2.7 – Transposes and Permutations

Course notes adapted from *Introduction to Linear Algebra* by Strang (5th ed), N. Hammoud's NYU lecture notes, and *Interactive Linear Algebra* by Margalit and Rabinoff, in addition to our text

LU Factorization

Example – Factorize (LU) the matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

Example – Factorize (LU) the matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

Example – Factorize (LU) the matrix A

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

Why is this Important?

Why are LU Factorizations Important?

Consider the system $A\boldsymbol{x} = \boldsymbol{b}$ with LU factorization $A = LU$. Then we have

$$L \underbrace{U\boldsymbol{x}}_{=\boldsymbol{y}} = \boldsymbol{b}.$$

Therefore we can perform (a now familiar) 2-step solution procedure:

1. Solve the lower triangular system $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$ by forward substitution.
2. Solve the upper triangular system $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$ by back substitution.

Moreover, consider the problem $AX = B$ (i.e., many different right-hand sides that are associated with the same system matrix). In this case we need to compute the factorization $A = LU$ only once, and then

$$AX = B \quad \Longleftrightarrow \quad LUX = B,$$

and we proceed as before:

1. Solve $LY = B$ by many forward substitutions (in parallel).
2. Solve $UX = Y$ by many back substitutions (in parallel).

Solving Systems of Equations

$$
\begin{aligned}
7x_1 - 2x_2 + x_3 &= 12 \\
14x_1 - 7x_2 - 3x_3 &= 17 \\
-7x_1 + 11x_2 + 18x_3 &= 5
\end{aligned}
$$

Solving Systems of Equations

Solving Systems of Equations

LDU Factorization

Goal

Let's try this with an Example

Find the LDU factorization of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

Let's try this with an Example – Find U

Find the LDU factorization of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

Let's try this with an Example – Find L

Find the LDU factorization of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

Let's try this with an Example – Find D

Find the LDU factorization of $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

Symmetric Matrices

What is a Symmetric Matrix?

Properties of a Symmetric Matrix

- The inverse of a symmetric matrix is symmetric.

Properties of a Symmetric Matrix

- AA^T and A^TA are symmetric.

Properties of a Symmetric Matrix

- If A is symmetric, then elimination leads to $A = LDL^T$.

Example

Factor the following symmetric matrices into $A = LDL^T$:

- $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$

Example

Factor the following symmetric matrices into $A = LDL^T$:

- $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Example

Factor the following symmetric matrices into $A = LDL^T$:

- $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

PA = LU

PA = LU?

So far, our assumption has been that we always have nonzero pivots. In case a pivot is zero, we can't use it to eliminate elements below it, so we have to exchange rows to find a nonzero pivot before we can start eliminating.

In this case, we can't find $A = LU$. We can, however, find $PA = LU$, where P corresponds to row exchanges done on A (in advance).

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

Find PA = LU for A below

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$