Final - Linear Algebra

Spring 2024 - Yiping

Name: Answer **NetID:** ____________

Please check ☑ your Professor's name:

- ☐ Professor Yiping Lu

While you wait, please read and check ☑ the following boxes:

- ☐ Unless I have extra time with the Moses Center, the time limit is **100 minutes**.
- ☐ I wrote my name and NetID (e.g. ab1234) at the top of this page.
- ☐ I will not detach any pages, especially not the scratch pages at the end.
- ☐ Except for multiple choice questions, I will show my work.
- ☐ If I need more space for an exercise, I will make a note and continue on one of the scratch pages.
- ☐ If I am caught in violation of academic integrity, including but not limited to peaking at another student's work, allowing another student to copy from my work, or speaking with another student, or using unauthorized resources, I will be asked to leave the exam and get a zero.

STOP

Do not start the exam until you are permitted to.

Exercise I [5 + 10 = 15 points]

Parts 1 and 2 are not related.

1. $A \in \mathbb{R}^{2\times 3}$ have singular value $\sqrt{2}, 0$, what is the eigenvalue of $A^\top A + I$. Is $A^\top A + I$ a positive definite matrix?
   (hint: What is the size of matrix $A^\top A + I$? How many eigenvalues you should have for this question?)

① $A = U\Sigma V^T$ (1')

$\Rightarrow A^TA + I = V\underline{\Sigma^T\Sigma V^T} + VV^T$ for V orthogonal

$= V(\Sigma^T\Sigma + I)V^T$

② $A^TA + I$ similar to $\Sigma^T\Sigma + I$ (1')

③ $\Sigma = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\Sigma^T\Sigma = \begin{pmatrix} 2 & & \\ & 0 & \\ & & 0 \end{pmatrix}$

$\Sigma^T\Sigma + I = \begin{pmatrix} 3^{(1')} & & \\ & 1^{(1')} & \\ & & 1^{(1')} \end{pmatrix}$

2. The determinant of the matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

equals 3. Find the determinant of the matrices below

$$B = 2 \times \underbrace{\begin{pmatrix} a & b & c \\ g & h & i \\ 2d-3g & 2e-3h & 2f-3i \end{pmatrix}}_{C}$$

Explain each answer briefly.

-48

$\det(B) = \underbrace{8}_{=2^3} \det(C)$

$= 8 \times \det\begin{pmatrix} a & b & c \\ g & h & i \\ 2d & 2e & 2f \end{pmatrix}$

$= 8 \times 2 \times \det\begin{pmatrix} a & b & c \\ g & h & i \\ d & e & f \end{pmatrix}$

$= \overset{4'}{8} \times \overset{3'}{2} \times \overset{3'}{-3}$

$= -48$

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There are more questions on this final starting on page 4.

Exercise II [6 + 3 + 3= 12 points]

1. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ -1 & -3 \\ -1 & -3 \end{bmatrix}$$

v_1 v_2

Use Gram-Schmidt to find the factorization $A = QR$.

2. Check $P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 \end{bmatrix}$ is the projection matrix that project to the column space of A.

3. What is the point in col(A) that is nearest to $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.

(1')　(2')

1. QR.
$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix}$$

$R = Q^T A$

$v_1 = u_1$

$u_2 = v_2 - \frac{u_2 \cdot u_1}{v_1 \cdot v_1} v_1$

(3')

2. $P = A(A^T A)^{-1} A$ (2')　Computation (1')

3. $\begin{bmatrix} 1.5 \\ 1.5 \\ 3.5 \\ 3.5 \end{bmatrix}$ (1')　$P.x$ (2')

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There are more questions on this final starting on page 6.

Exercise III [6 + 4 = 10 points]

*hint: Parts 1 and 2 are **related**.*

1. P is a Projection Matrix, prove P is symmetric and $P^2 = P$.
   hint: *If my matrix P is project to the column space of A, what is my matrix P? Can you compute $P^\top$ and P^2.*

$P = A(A^TA)^{-1}A^T$ (2')

$P^T = (A^T)^T((A^TA)^{-1})^T A^T$

$= A(A^TA)^{-1}A^T$ A^TA symmetric

$= P$ (2')

$P^2 = \underbrace{A(A^TA)^{-1}}A^T\underbrace{A(A^TA)^{-1}A^T}$

$= A(A^TA)^{-1}A^T$ (2')

2. Prove that $I - 2P$ is an orthogonal matrix
   hint: *compute $(I-2P)^\top(I-2P)$ using $P^\top = P$ and $P^2 = P$*

$(I-2P)^T(I-2P) = (I-2P)(I-2P)$ (2') $P = P^T$

$= I - 4P + 4P^2$ (2')

$= I - 4P + 4P = I$

Exercise IV (5 + 3 + 2 + 3 = 13 points)

Consider the following matrices

Compute eigen (1)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

1. What are the dimensions of the eigenspaces in each case?

2. Which one can be diagonalized?

eigen 0	2	1	1	(2, 1)
1	1	1	1	(2, 1)
diag	✓	×	×	(3, 1)

3. Check $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are similar by using $X = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

4. Compute A^{100}. (*hint*: try to use question 3)

3. Check $A = X B X^{-1}$ (2')

4. $A^{100} = X B^{100} X^{-1}$ (2')

$$B^{100} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$ (1')

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Exercise V (6 + 8 + 6 = 20 points)

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. The goal of this question is to compute the singular value decomposition $A = U\Sigma V^T$. You must complete this exercise in the given order correctly to receive full credit.

1. First, find A^TA and its eigenvalue(s). (6') 2.1
2. Then, find the matrix V. (8')
3. Finally, find the matrix U and then write the full SVD.

2'

$$\underbrace{\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}}_{u} \underbrace{\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{V}$$

$u_1 = \frac{1}{\sigma_1} A v$ (2')

$u_2 = \frac{1}{\sigma_2} A v$ (2')

$u_3 = \underline{AA^T}$ Nul space. (2')

Continue Exercise V here if needed.

Exercise VI ($2 \times 5 = 10$ points)

The SVD of a certain matrix M is below. From this SVD, give the following:

$$\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix}$$

1. The size of M (explain briefly)

2×3

2. The rank of M (explain briefly)

1

3. A basis for the Column space of M.

$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

4. A basis for the Nul space of M.

$\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{2} \\ 1/\sqrt{3} \end{bmatrix}$

5. A basis for the Row space of M.

$\begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

Exercise VII (10 + 5 = 15 points)

Consider the two transformations below:

- $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T((x_1, x_2)) = (x_1 + x_2, x_1 - 2x_2, 3x_1 + 1)$
- $S : \mathbb{P}_2 \to \mathbb{P}_3$ defined by $S(p(x)) = 3xp(x+1)$

One is linear, prove this fully, try to write down the transformation matrix for it (10 points+5 extra credits). The other is not linear, explain why (a counterexample suffices, 5 points).

① Not (5)

② $S(P_1 + P_2) = 3x(P_1 + P_2)(x+1)$

$= 3x P_1(x+1) + 3x P_2(x+1)$

$= S(P_1) + S(P_2)$ (5')

$S(cP_1) = 3x(cP_1)(x+1)$ (5')

$= c\,3xP_1(x+1)$

$= c\,S(P_1)$ (5')

Matrix

$P = ax^2 + bx + c$

$$\begin{bmatrix} 3 & 0 & 0 \\ 6 & 3 & 0 \\ 3 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$S(p) = 3x(a(x+1)^2 + b(x+1) + c)$

$= 3x(ax^2 + (2a+b)x + (a+b+c))$

$= 3ax^3 + (6a+3b)x^2 + (3a+3b+3c)$

Exercise VIII (5 points)

Answer **one** of the two below. If you submit both, your score will be their average.

1. Suppose $Q = \begin{bmatrix} 2 & 3 \\ 3 & -6 \\ 6 & 2 \end{bmatrix}$. Consider the linear transformation:

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \quad f(v) = Qv$$

   Suppose you have any two orthogonal (i.e., perpendicular) vectors $v_1, v_2 \in \mathbb{R}^2$. Use linear algebra to prove that the vect $f(v_1), f(v_2) \in \mathbb{R}^3$ are also orthogonal.

2. For $A \in \mathbb{R}^{n\times n}$ has eigenvalue λ, prove $A^2 + 2A + 3I$ have eigenvalue $\lambda^2 + 2\lambda + 3$.

   hints: they have the same eigenvectors.

1. $f(v_1)^T f(v_2) = v_1 \underbrace{Q^T Q}_{= \begin{bmatrix} 19 & 0 \\ 0 & 19 \end{bmatrix}} v_2 = 19 \, v_1^T v_2 = 19 \times 0$

$= 0$

2. $Ax = \lambda x$

$\Rightarrow (A^2 + 2A + 3I)x = (\lambda^2 + 2\lambda + 3)x$

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