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NEWYORK UNIVERSITY

Linear Algebra Final Review

Subject: MATH-UA 140 Linear Algebra                Year: 2024(Sem 2)

Name of Examiners: .                Time allow: 1001

Instruction to Candidate : (only on page 1)

(1) This paper contains ________ questions.

(2) Candidates must answer ________ questions.

Question No 1

Diagonalize $A$ and compute $V\mathbf{A}kV^{-1}$ to prove this formula for $A^k$:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \text{ has } A^k = \frac{1}{2}\begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix}.$$

and what is the meaning of $\lim_{k\to\infty} \frac{1}{3^k} A^k$. $= \left(\frac{A}{3}\right)^k$.

## Solution:

↳ $\lambda_1 = 1$ $v_1 = (-1, 1)$

↳ $\lambda_2 = 1/3$ $v_2 = (1, 1)$

The eigenvalues of $A$ are 3 and 1, and the corresponding eigenvectors are $v_1 = (-1, 1)$, $v_2 = (1, 1)$. Therefore, $A$ can be diagonalized as $A = V\Lambda V^{-1}$, where $V = [v_1, v_2]$, $\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ and $V^{-1} = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. $A^k = V\Lambda^k V^{-1} = \frac{1}{2}\begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix}$.

$\lim_{k\to\infty} \frac{1}{3^k} A^k = \lim_{k\to\infty} \frac{1}{2}\begin{pmatrix} \frac{1}{3^k}+1 & \frac{1}{3^k}-1 \\ \frac{1}{3^k}-1 & \frac{1}{3^k}+1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (the largest eigen vector)

Examination Requirements: NIL

$\frac{1}{2}v_1$   $-\frac{1}{2}v_1$ → eigenvector of largest eigenvalue

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Householder Matrix

Question No 2

$P = uu^T$ is a projection

For $u$ is a unit vector prove that $Q = I - 2uu^\top$ is an symmetric orthogonal matrix. Prove $\|Qx\| = \|x\|$. (*)

$Q^TQ = QQ^T = I$

**Solution:**

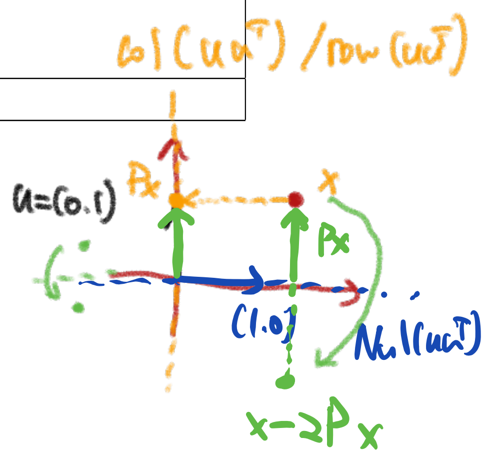
$Q$ is symmetric because $uu^\top$ is symmetric.
Then $QQ^\top = QQ = Q^2 = (I - 2uu^\top)^2 = I - 2uu^\top - 2uu^\top + 4u \underbrace{u^\top u}_{u^\top u = \|u\|^2 = 1} u^\top = I - 4uu^\top + 4uu^\top = I$

For all orthogonal matrix $Q$, we have

$$\|Qx\|^2 = (Qx)^\top(Qx) = x^\top Q^\top Qx = x^\top x = \|x\|^2$$

for $Q^\top Q = I$

Examination Requirements: NIL




Example. $\vec{u} = (0,1)$

$P\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\y\end{pmatrix}$

$P = \vec{u}\vec{u}^T = \begin{pmatrix}0\\1\end{pmatrix}(0, 1) = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$

$I - 2\vec{u}\vec{u}^T = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

$(I - 2uu^T)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\-y\end{pmatrix}$

$(I - 2P)x = x - 2Px$

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Question No 3

What are the four fundamental subspaces of $M = I - P$ in terms of the column space of $P$.

## Solution

For a projection matrix $P$: (projection matrix is always symmetric)

- $x \in \mathrm{col}(P) = \mathrm{row}(P)$: Px=x
- $x \in \mathrm{Nul}(P) = \mathrm{LeftNul}(P)$: Px=0

For matrix $I - P$

- $x \in \mathrm{col}(P) = \mathrm{row}(P)$: (I-P)x=x-Px=x-x=0
- $x \in \mathrm{Nul}(P) = \mathrm{LeftNul}(P)$: (I-P)x=x-Px=x-0=x

is a also a projection matrix.
Left Null space = Right Null space = Colume space of $P$.
Column space = Row space = orthogonal complement of the colume space of $P$.

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Question No 4

$P$ is a Projection Matrix, prove $P$ is symmetric and $P^2 = P$. What is the eigenvalue of Projection matrix $P$. Prove that $I - 2P$ is an orthogonal matrix

## Solution

$P = A(A^TA)^{-1}A^T$ then

- $P^2 = A\underbrace{(A^TA)^{-1}A^TA}_{=I}(A^TA)^{-1}A^T = A(A^TA)^{-1}A^T = P$
- $P^T = (A(A^TA)^{-1}A^T)^T = A^T(A^TA)^{-T}A = A(A^TA)^{-1}A^T$ (For $A^TA$ symmetric)

Eigenvalue is $1, 0$ (for $P^2 = P$ so eigenvalues should satisfies $\lambda^2 = \lambda$)
Since $P$ is a projection matrix, we have $P = P^T$. To show that $Q$ is an orthogonal matrix, we need to check that $QQ^T = I$. We have

$$QQ^T = (I - 2P)(I - 2P)^T$$

.

$$= (I - 2P)(I^T - 2P^T)$$

$$= (I - 2P)(I - 2P) \quad \text{(since } I \text{ and } P \text{ are symmetric)}$$

$$= I - 4P + 4P^2$$

Since for a projection matrix we have $P^2 = P$, this product is equal to $QQ^T = I$, as required.

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Question No 5

If $A^2 = -A$, what is the possible value of $\det(A)$.

## Solution

$A^2 = -A$ means $\det(A^2) = \det(-A)$ however

- $\det(A^2) = \det(A)^2$ $\det(AB) = \det(A)\det(B)$ $\det(c \cdot A) = c^n A$
- $\det(-A) = (-1)^n \det(A) = \begin{cases} -\det(A) & \text{if } n \text{ is odd} \\ \det(A) & \text{if } n \text{ is even} \end{cases}$ $A \in \mathbb{R}^{n \times n}$

Thus

$$\det(A)^2 = \begin{cases} -\det(A) & \text{if } n \text{ is odd} \\ \det(A) & \text{if } n \text{ is even} \end{cases}$$

which means

$$\det(A) = \begin{cases} 0, -1 & \text{if } n \text{ is odd} \\ 0, 1 & \text{if } n \text{ is even} \end{cases}$$

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Question No 5

Suppose an $m \times n$ matrix $A$ has rank $r$. What are the ranks of

(a) $A^T$?

(b) $AA^T$?

(c) $AA^T + \lambda I$ $(\lambda > 0)$?

(d) $A^TAA^T$?

**Solution**

**Answer 1**

(A) $r$

(B) we showed in class it's $r$ (page 17 in `https://2prime.github.io/files/linear/linearslide14filled.pdf`)

(C) it's a positive definite matrix with all eigenvalues lareger than $\lambda$, think why.

(D) $r$ (similar page 17 in `https://2prime.github.io/files/linear/linearslide14filled.pdf`)

**Answer 2 Using SVD**

(A) $\text{rank}(A^T) = \dim(\text{row}(A^T)) = \dim(\text{col}(A)) = \text{rank}(A) = r$.

(B) Let $A = U\Sigma V^T$ be a full SVD. Then,

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma^2 U^T.$$

Thus, $U\Sigma^2 U^T$ is a SVD of $AA^T$. If $\Sigma$ has $r$ positive singular values then so will $\Sigma^2$. Therefore, the rank of $AA^T$ is $r$.

(C) Since $I_m = UU^T$, the equation above yields $AA^T + \lambda I = U\Sigma^2 U^T + \lambda I = U(\Sigma^2 + \lambda I_m)U^T$. Since $\Sigma^2 + \lambda I = \text{diag}(\sigma_1^2 + \lambda, \ldots, \sigma_r^2 + \lambda, \ldots, \lambda)$, the rank is $m$.

(D) $A^TAA^T = (U\Sigma V^T)^T(U\Sigma V^T)(U\Sigma V^T)^T = V\Sigma^T U^T U\Sigma V^T U^T U\Sigma V^T = V\Sigma^T\Sigma\Sigma^T V^T = V\Sigma^3 V^T$. $\Sigma^3$ has $r$ positive singular values as like $\Sigma$. Therefore, the rank is $r$.

Examination Requirements: NIL

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$\begin{bmatrix} 1 & 1 & \\ 0 & 1 & \\ & & 2 \end{bmatrix}$ can't be diag    $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 2 \end{bmatrix}$ diag

Question No 6

The following matrices have only one eigenvalue: 1. What are the dimensions of the eigenspaces in each case?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

For a matrix $A$, the eigenspace with eigenvalue $\lambda$ is the kernel of the matrix $A - \lambda I$. Here we have $\lambda = 1$, so we subtract $I$ from each of the matrices above:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and find the dimensions of the kernels.

The ranks of these matrices are 0, 2, 2, 1 respectively, so by the rank-nullity theorem the dimensions of the kernels are 3, 1, 1, 2.

**Answer:** 3, 1, 1, 2.

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Question No 7

For $A \in \mathbb{R}^{n\times n}$ has singular value $\sigma_1, \cdots, \sigma_n$ prove

$(\Sigma^T\Sigma + \lambda I)^{-1} \to \begin{bmatrix} \frac{1}{\sigma_1^2+\lambda} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2+\lambda} \end{bmatrix}$

$\parallel$

$\begin{bmatrix} \sigma_1^2+\lambda & & \\ & \ddots & \\ & & \sigma_n^2+\lambda \end{bmatrix}$

- $\mathrm{tr}(A^\top A) = \sigma_1^2 + \cdots + \sigma_n^2$
- $\mathrm{tr}((A^\top A + \lambda I)^{-1} A^\top A) = \frac{\sigma_1^2}{\sigma_1^2+\lambda} + \cdots + \frac{\sigma_n^2}{\sigma_n^2+\lambda}$

Very similar to $A(A^TA)^{-1}A^T$

## Solution

$\Sigma^T\Sigma = \begin{pmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{pmatrix}$

Using SVD $A = U\Sigma V^\top$ Then we have

= similar 1

- $A^\top A = V\Sigma^\top\Sigma V^\top$ so $\mathrm{tr}(A^\top A) = \mathrm{tr}(\Sigma^\top\Sigma) = \sigma_1^2 + \cdots + \sigma_n^2$
- $(A^\top A + \lambda I) = V\underbrace{(\Sigma^\top\Sigma + \lambda I)}_{\text{diag}}V^\top$, $(A^\top A + \lambda I)^{-1} = V\underbrace{(\Sigma^\top\Sigma + \lambda I)^{-1}}_{\text{diag}}V^\top$
- $(A^\top A + \lambda I)^{-1}A^\top A = V(\Sigma^\top\Sigma + \lambda I)^{-1}\Sigma^\top\Sigma V^\top = V \begin{bmatrix} \frac{\sigma_1^2}{\sigma_1^2+\lambda} & 0 & \cdots & 0 \\ 0 & \frac{\sigma_2^2}{\sigma_2^2+\lambda} & \cdots & 0 \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \frac{\sigma_n^2}{\sigma_n^2+\lambda} \end{bmatrix} V^\top$
- $$\mathrm{trace}((A^\top A+\lambda I)^{-1}A^\top A) = \mathrm{trace}\left(\begin{bmatrix} \frac{\sigma_1^2}{\sigma_1^2+\lambda} & 0 & \cdots & 0 \\ 0 & \frac{\sigma_2^2}{\sigma_2^2+\lambda} & \cdots & 0 \\ \cdots & \cdots & \cdots & \\ 0 & 0 & \cdots & \frac{\sigma_n^2}{\sigma_n^2+\lambda} \end{bmatrix}\right) = \frac{\sigma_1^2}{\sigma_1^2+\lambda} + \cdots + \frac{\sigma_n^2}{\sigma_n^2+\lambda}$$

Examination Requirements: NIL

# Projection.

$A\vec{x} = \vec{b}$ **Question** What is the nearest point of $\vec{b}$ in $Col(A)$

$A\vec{x} \in Col(A)$

**answer**. projection Matrix

$$P = A(A^TA)^{-1}A^T$$

**behind**. $A\vec{x} = \vec{b} \Rightarrow A^TA\vec{x} = A^T\vec{b} \Rightarrow \vec{x} = (A^TA)^{-1}A^T\vec{b}$ least square.

$\Downarrow$

$$A\vec{x} = \underbrace{A(A^TA)^{-1}A^T}_{\text{projection matrix}}\vec{b}$$

## —. Hints (Exercise 2-4)

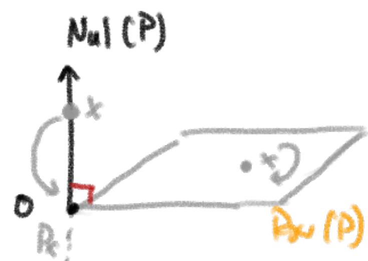
① $\underline{\|u\|=1}$. $P = u(\underbrace{u^Tu}_{u^Tu=1})^{-1}u^T = uu^T$ $\|u\|=1$ $u^Tu = \|u\|^2 = 1$ rank1 matrix. projection

② P is a Projection Matrix. $P^2 = P$ $P = P^T$

$$P^2 = A(A^TA)^{-1}\underbrace{A^TA(A^TA)^{-1}}_{=I}A^T = A(A^TA)^{-1}A^T = P$$

$P = A(A^TA)^{-1}A^T$ $\rightarrow Col(P)$

③ $P^2 = P$ means $\underbrace{P.(Px)}_{\text{projection 2 Times}} = \underbrace{Px}_{\text{Projection 1 time}}$.




row(P) $=$

— $x \in Col(P)$ $Px = x$. x is eigenvertor with eigenvalue 1

$Nul(P) \perp Row(P)$

— $x \in Nul(P)$ $Px = 0$

$=$ left $Nul(P)$ x is eigenvector with eigenvalue 0.

④ P only have eigenvalue 1 or 0

$P^2 = P \Rightarrow P^2 - P = 0 \Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \lambda = 1, 0$

△ If $\lambda$ is the eigenvalue of P, Then $\lambda^2 - \lambda$ is the eigen $P^2 - P$

$Px = \lambda x \Rightarrow (P^2 - P)x = P^2x - Px = (\lambda^2x - \lambda x) = (\lambda^2 - \lambda)x$

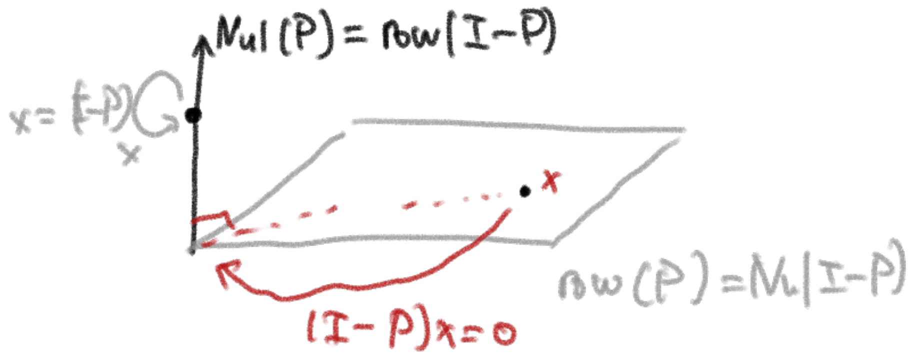
⑤ $P = \underbrace{u_1u_1^T + u_2u_2^T + \cdots + u_ru_r^T}_{} + \underbrace{0 \cdot u_{r+1}u_{r+1}^T + \cdots + 0\, u_nu_n^T}_{}$

$u_1 \cdots u_r$ is the orthonormal basis of row(P)/Col(P) $u_{r+1} \cdots u_n$ is the orthonormal basis of Nul(P)/left Nul(P)

(6) $I = u_1u_1^T + u_2u_2^T + \cdots + u_ru_r^T + u_{r+1}u_{r+1}^T + \cdots + u_nu_n^T$

$u_1 \cdots u_n$ are $I$'s eigenvector with eigenvalue 1

$I - P = \underbrace{0\cdot u_1u_1^T + 0\cdot u_2u_2^T + \cdots + 0 u_ru_r^T}_{u_1, u_2, \cdots u_r \text{ is the orthonormal basis of } Nul(I-P) / \text{left } Nul(I-P)} + \underbrace{u_{r+1}u_{r+1}^T + \cdots + u_nu_n^T}_{u_{r+1} \cdots u_n \text{ is the orthonormal basis of } Col(I-P) / Row(I-P)}$.

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Gram - Schmidt Process.

- $\{x_1 \cdots x_n\}$ basis $\Rightarrow$ $\{u_1 \cdots u_n\}$ orthogonal.

Hint. Orthogonal matrix → Column is orthonormal

If you want orthogonal, you need to normalize your vector to unit vector!

- QR Decomposition. $R = Q^TA$

$A = QR$, $A^TA = R^T\underbrace{Q^TQ}_{I}R = R^T R$
lower upper

But it's not LU Decomposition!
↓
LU need diag of $L$ to be 1.

# Diagonalization & Eigen Vectors,

$A = X \Lambda X^{-1}$ $\quad A \in \mathbb{R}^{n \times n}$

- $X = [x_1 \cdots x_n]$ $\quad x_1 \cdots x_n$ are linear independent Eigen vectors

This is not always true.

- In the case, $\lambda_1, \lambda_2, \cdots \lambda_n$ are all different numbers, this is true.
(Excise 6)

- Example. $A = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$ $\quad p(\lambda) = (\lambda - 1)^2$ $\quad \lambda_1 = 1 \quad \lambda_2 = 1$

$\Rightarrow A - I = \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} \rightarrow$ 1 dimension eigen space.

- $A^k = X \Lambda^k X^{-1}$ $\quad$ Excise 1

- $A = \mathbb{X} B \mathbb{X}^{-1}$ similar matrix

same eigenvalue. $\rightarrow$ different eigenvectors.

same trace

same det.

$Ax = \lambda x$

$\mathbb{X} B \mathbb{X}^{-1} x = \lambda x$

$\Rightarrow B \mathbb{X}^{-1} x = \lambda \mathbb{X}^{-1} x$

$\Rightarrow \mathbb{X}^{-1} x$ is the eigen

$\Downarrow$

hint us the way to compute the matrix $\mathbb{X}$. (If we know A and B)

# Symmetric Matrix

- Eigenvalue is real, can always be diagonalized
  Eigenvectors is orthogonal to each other.
  $Q^T = Q^{-1}$
- $A = Q \Lambda Q^T$ $\Rightarrow$ A is similar to $\Lambda$
  Q is an orthogonal matrix
  eigenvalue eigenvector
- $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$
  rank 1
  symmetric
  Projection Matrix.

# SVD

Any Matrix $A \in \mathbb{R}^{m \times n}$

- $\underset{m\times n}{A} = \underset{m\times m}{U} \underset{m\times n}{\Sigma} \underset{n\times n}{V^T}$
  orth diag orth

symmetric $AA^T = U \Sigma \Sigma^T U^T$, U : eigenvectors of $AA^T$

symmetric $A^TA = V \Sigma^T \Sigma V^T$, V : eigenvector of $A^TA$

- $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T$

$\Rightarrow$ U, V can provide orthonormal basis of the Four fundamental subspace of A.

- How to compute SVD
  - Use. $U$ : eigenvector of $AA^T$ or
    $V$ : eigenvector of $A^TA$
  - use
    $u_1 = \frac{1}{\sigma_1} A v_1$ or $v_1 = \frac{1}{\sigma_1} A^T u_1$

- Properties of Det
- Cofactor.
- Check if a Transform is linear Transform (HW6)

  △ it is linear Transform
  ⇒ prove by checking
  - $T(cv) = c\,T(v)$ ← $c=0$, $T(0)=0$
  - $T(v_1+v_2) = T(v_1) + T(v_2)$

  ○ It's not a linear Transform
  give a counterexample.

$Px = \lambda x$

$PPx = P(\lambda x) = \lambda Px = \lambda \lambda x = \lambda^2 x$

Orthogonal Matrix $Q \in \mathbb{R}^{n \times n}$ square matrix

$$Q^TQ = I \iff QQ^T = I$$

both means $Q^T = Q^{-1}$

Orthogonal Matrix $Q \in \mathbb{R}^{n \times m}$

means $Q$ : orthonormal basis

$Q^TQ$ $m \times m$ matrix $= I$ eigen $\underbrace{1 \cdots 1}_{m}$

$\Downarrow$

$QQ^T$ $n \times n$ matrix eigen $\underbrace{1 \cdots 1}_{n}$ $\underbrace{0 \cdots 0}_{m-n}$

means $QQ^T$ is projection!

$Ax = b \Rightarrow \underbrace{A^TA}_{\text{square matrix}}x = A^Tb \Rightarrow x = (A^TA)A^Tb$ least square solution

$Ax = \underline{A(A^TA)^{-1}A^T}b$

projection matrix

$P = A(A^TA)^{-1}A^T$

- $Px$ is the nearest point to $b$ in $Col(A)$

this means. $x - Px \perp Col(A)$

$x - Px \perp Col(P) \Rightarrow x - P_x = leftNul(P)$

Exercise 2 - Exercise

$Col(I-P) = leftNul(P)$

---

$A = [v_1, v_2]$, $\vec{u}$ is orthogonal to $\vec{v_1}, \vec{v_2}$

$\vec{u}^T\vec{v_1} = 0$

$\vec{u}^T\vec{v_2} = 0$

$\Rightarrow u^T\underbrace{[v_1, v_2]}_{A} = [\vec{u}^T\vec{v_1}, \vec{u}^T\vec{v_2}] = [0, 0]$

$\Rightarrow A^Tu = 0$, $u \in leftNul(A)$

---

① $\|u\| = 1$, Projection Matrix to $u$

$P = u\underbrace{(u^Tu)^{-1}}_{\|u\|=1}u^T = uu^T$ rank 1 Symmetric Matrix

② $P$ is also symmetric. $P^T = (A(A^TA)^{-1}A^T)^T = (A^T)^T(A^TA)^{-T}A^T$

$= A(A^TA)^{-T}A^T = A(A^TA)^{-1}A^T$

$A^TA$ is symmetric

③ $P^2 = P$

$\Uparrow$

$y \in Col(P) = Px$

Project($y$) to $P$. the answer is $y$

$P(Px) = P^2x = Px$

proof. $P^2 = A(A^TA)^{-1}\underbrace{A^TA(A^TA)^{-1}}_{I}A^T = A(A^TA)^{-1}A^T = P$

④ $P$ has eigenvalue 1 and 0. $\lambda$ is $P$'s eigenvalue

$Px = \lambda x = P^2x = \lambda^2x \Rightarrow \lambda^2 = \lambda \Rightarrow \lambda = 0, 1$

⑤ Fact

rank $(P) = r$

$x \in Col(P)$ $\quad Px = x \quad$ eigenvalue 1 $\quad \vec{u}_1 \cdots \vec{u}_r$ (orthonormal)

$x \in Nul(P)$ $\quad Px = 0 \quad$ eigenvalue 0 $\quad \vec{u}_{r+1} \cdots \vec{u}_n$ (orthonormal)

$$\Rightarrow P = 1\,\vec{u}_1\vec{u}_1^T + 1\,\vec{u}_2\vec{u}_2^T + \cdots + 1\cdot\vec{u}_r\vec{u}_r^T + 0\cdot\vec{u}_{r+1}\vec{u}_{r+1}^T + \cdots + 0\,\vec{u}_n\vec{u}_n^T.$$

$I - P$

$x \in Col(P)$ $\quad (I-P)x = x - Px = 0 \quad$ eigenvalue 0 $\quad \vec{u}_1 \cdots \vec{u}_r$

$x \in Nul(P)$ $\quad (I-P)x = x - Px = x \quad$ eigenvalue 1 $\quad \vec{u}_{r+1} \cdots \vec{u}_n$

$$I - P = 0\cdot\vec{u}_1\vec{u}_1^T + 0\cdot\vec{u}_2\vec{u}_2^T + \cdots + 0\cdot\vec{u}_r\vec{u}_r^T + 1\cdot\vec{u}_{r+1}\vec{u}_{r+1}^T + \cdots + 1\,\vec{u}_n\vec{u}_n^T$$

$$I = P + (I-P)$$

$$= \vec{u}_1\vec{u}_1^T + \vec{u}_2\vec{u}_2^T + \cdots + \vec{u}_r\vec{u}_r^T + \vec{u}_{r+1}\vec{u}_{r+1} + \cdots + \vec{u}_n\vec{u}_n^T.$$

Col. $P$ is a Projection $\Rightarrow I - P$ is also Projection.

$Col(P) = Nul(I-P) \Rightarrow \vec{u}_1 \cdots \vec{u}_r$

$Nul(P) = Col(I-P) \Rightarrow \vec{u}_{r+1} \cdots \vec{u}_n$

Eigenvector

$Ax = \lambda x$

$A \in \mathbb{R}^{n\times n}$

$\lambda \in \mathbb{R}$. Eigenvalue.

- $\lambda$ is the solution of $P(\lambda) = \det(\lambda I - A)$

  $x \in \text{Nul}(A - \lambda I)$

- Diagonalization. $A = X \Lambda X^{-1}$ diag.

  $X = [\vec{x}_1, \vec{x}_2, \dots \vec{x}_n]$ $\vec{x}_1, \vec{x}_2 \cdots \vec{x}_n$ linear independ eigenvectors.

  This will not alway exist!

① $\lambda_1, \lambda_2, \cdots \lambda_n$ are different eigen $\Rightarrow \vec{x}_1, \vec{x}_2, \dots \vec{x}_n$ linear indepen

$\Rightarrow$ A can be diag

② Example $A = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$ $P(\lambda) = (\lambda - 1)^2 \rightarrow \lambda_1 = \lambda_2 = 1$

$\Rightarrow \text{Nul}(A - I) = \text{Nul}\left(\begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}\right)$ is only 1 dimension space

A can't be digonalized.

---

$A = X B X^{-1}$ Similar.

- same trace
- same determinante
- same eigenvalues. x eigenvector

x is A's eigenvector

$\Rightarrow X^{-1}x$ is B's eigenvector

$Ax = \lambda x$

$X B X^{-1} x = \lambda x$

$\Rightarrow B X^{-1} x = \lambda X^{-1} x$

---

$A = X \Lambda X^{-1} \Rightarrow A^k = X \Lambda^k X^{-1}$ Exercise!

Symmetric

$A = Q \Lambda \underline{Q}^T$ $\quad Q^{-1}$ $\quad Q$ orthogonal

- Symmetric is diagonalizable, eigenvalues is real.
  the eigenvector of A are orthogonal to each other.

$Q = [u_1 \cdots u_n]$ $\quad u_1 \cdots u_n$ are eigenvector
$\|u_1\| = \cdots \|u_n\| = 1$ (orthonormal)

- $A = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$

rank 1 symmetric
projection matrix
project to $u_1$

SVD All Matrix $\mathbb{R}^{m\times n}$

$$\underset{m\times n}{A} = \underset{m\times m}{U}\underset{m\times n}{\Sigma}\overset{n\times n}{V^T}$$

U. V. square orthogonal matrix

$\Sigma$ diag

symmetric

- $A^TA = V\Sigma^T\Sigma V^T$

symmetric

$AA^T = U\Sigma\Sigma^TU^T$

$V: [\vec{v}_1 \cdots \vec{v}_n]$ $\vec{v}_1 \cdots \vec{v}_n$ eigenvector of $A^TA$

$U: [\vec{u}_1, \cdots \vec{u}_n]$ $\vec{u}_1 \cdots \vec{u}_n$ - - - - - of $AA^T$

$Q$ : orthogonal matrix

$Q \in \mathbb{R}^{n \times n}$ square matrix

$Q^T Q = Q Q^T = I$

$\|u_i\| = 1$ !!

$Q = [\vec{u}_1, \vec{u}_2, \dots \vec{u}_n]$ ($\vec{u}_1, \dots \vec{u}_n$ are orthonormal)

---

Find Orthogonal Basis $\leftarrow$ G-S

it's not orthonormal

$\swarrow$ normalize to unit vector

$A = \underline{Q} \, R$

orthogonal upper triangular

$\Rightarrow R = Q^T A$

**Remark** $A^T A = (QR)^T Q R = R^T \underbrace{Q^T Q}_{I} R = R^T R$

Lower upper

However it's not CU. let LU diag needs to be 1.