Linear Algebra – Final Exam Review

1 True or False?

1. If $\vec{v_1}$ and $\vec{v_2}$ are linearly independent e-vectors of A, then they correspond to different e-values.
2. $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if there is a nonzero solution to $(A - \lambda I)\vec{x} = 0$.
3. To find the eigenvectors of A, we reduce the matrix A to row echelon form.
4. Any invertible matrix is diagonalizable.
5. Two matrices that have the same eigenvalues (with same multiplicities) should be similar.
6. If A (square matrix) is diagonalizable and invertible, then (choose the right answer)
   - Its inverse is diagonalizable and is invertible
   - Its inverse is diagonalizable and could be invertible
   - Its inverse could be diagonalizable and is invertible
   - Its inverse could be diagonalizable and could be invertible
   - We need more information to determine this

2 Some Proofs

1. Suppose a square matrix A is such that its square is zero. Find all eigenvalues of A.
2. We say that a matrix M is idempotent if $M^n = M$ for all positive integers n. Let A and B be two similar matrices. Show that if one of those matrices is idempotent, then the other is.
3. Let A be an $n \times n$ matrix with determinant 5 . What is the determinant of $-A$?
4. Let A be an orthogonal 2×2 matrix. Show that $\|A\vec{x}\| = \|\vec{x}\|$ for all vectors $\vec{x}$ in $\mathbb{R}^2$. Is this result generalizable (for any $n \times n$ matrix)?

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① $A \in M_{n \times n}$ st. $A^2 = 0$.

e-value satisfies $A A\vec{x} = A\lambda\vec{x}$ for some $\vec{x} \neq \vec{0}$

$A^2\vec{x} = \lambda(A\vec{x})$

But $A^2 = 0$; so $0\vec{x} = \lambda(A\vec{x})$; $\vec{0} = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}$

$A\vec{x} = \lambda\vec{x}$

But $\vec{x} \neq \vec{0}$; $\lambda^2 = 0$

$\boxed{\lambda = 0}$

② A and B are similar: $\exists P \mid B = PAP^{-1}$

A idempotent $\Rightarrow A^n = A \quad \forall n \in \mathbb{N}$

B idempotent $\Rightarrow B^n = B \quad \forall n \in \mathbb{N}$.

scratch: suppose B idem. Then $B^2 = B$

$\underbrace{(PAP^{-1})^2}_{B^2} = (PAP^{-1})(PAP^{-1}) = PA(P^{-1}P)AP^{-1} = \underbrace{PA^2P^{-1}}$

we know: $B^2 = B \Rightarrow PA^2P^{-1} = PAP^{-1}$

$\Rightarrow \dots \Rightarrow A^2 = A$

Pf/ let B be idempotent ($B^n = B \ \forall n \in \mathbb{N}$) and $B = PAP^{-1}$ $\exists P$ (so A and B are similar).

Want To Show: $A^n = A \quad \forall n \in \mathbb{N}$.

$$B^n = B, \quad B = PAP^{-1} \quad \Big\} \quad (PAP^{-1})^n = PAP^{-1}$$

$$\underbrace{(PAP^{-1})(PAP^{-1})\dots(PAP^{-1})}_{n \text{ times}} = PAP^{-1}$$

$$PA\underbrace{(P^{-1}P)}_{I}PA\underbrace{(P^{-1}P)}_{I}\dots\underbrace{(P^{-1}P)}_{I}AP^{-1} = PAP^{-1}$$

$$P^{-1}PA^nP^{-1}P = P^{-1}PAP^{-1}P$$

$$\boxed{A^n = A}$$ Thus A is idempotent

3. let $A \in M_{n\times n}$ with $|A| = 5$. Find $|-A|$?

ex: $A = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$ $\quad |A| = 5$

$-A = \begin{bmatrix} -* & -* \\ -* & -* \end{bmatrix}$; $|-A| = \underbrace{(-1)(-1)}_{\text{each row scaled by } -1}(5) = 5$

but if A has 3 rows, we get $|-A| = (-1)^3 5 = -5$

nbr of rows

$$|-A| = (-1)^n |A| = \begin{cases} |A| & \text{if } n \text{ even} \\ -|A| & \text{if } n \text{ odd} \end{cases}$$

for this Matrix; we don't know the exact number of rows, so that's the best we can do

3. We say that a matrix M is idempotent if $M^n = M$ for all positive integers n. Let A and B be two similar matrices. Show that if one of those matrices is idempotent, then the other is.

4. Let A be an $n \times n$ matrix with determinant 5. What is the determinant of $-A$?

5. Let A be an orthogonal 2×2 matrix. Show that $\|A\vec{x}\| = \|\vec{x}\|$ for all vectors $\vec{x}$ in $\mathbb{R}^2$. Is this result generalizable (for any $n \times n$ matrix)?

3. ($M^n = M$) A and B are similar matrices, and without loss of generality, assume A is idempotent; so $A^n = A$ (Goal: $B^n = B$). Since A and B are similar, there exists some matrix P such that $B = PAP^{-1}$

scratch: $n = 2$: $A^2 = A$ (given)

$$B^2 = \underbrace{(PAP^{-1})}_{B}\underbrace{(PAP^{-1})}_{B} = PA\underbrace{(P^{-1}P)}_{I}AP^{-1} = PA^2P^{-1}$$

$$B^2 = \underbrace{PAP^{-1}}$$

so $B^2 = B$

$$B^n = \underbrace{(PAP^{-1})(PAP^{-1}) \dots (PAP^{-1})}_{n \text{ times}}$$

$$B^n = PA\underbrace{(P^{-1}P)}_{I}A\underbrace{(P^{-1}P)}_{I} \dots \underbrace{(P^{-1}P)}_{I}AP^{-1} = P\underbrace{A \cdot A \cdot A \dots A}_{n \text{ times}}P^{-1}$$

so $B^n = PA^nP^{-1}$; since $A^n = A$ (A is idempotent)

we get $B^n = PAP^{-1} = B$

4. Let A be an orthogonal 2×2 matrix. Show that $\|A\vec{x}\| = \|\vec{x}\|$ for all vectors $\vec{x}$ in $\mathbb{R}^2$. Is this result generalizable (for any $n \times n$ matrix)?

A is orthogonal $\Rightarrow$ $A^T = A^{-1} \Leftrightarrow A^T A = I$

Recall: $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2} = \sqrt{\underbrace{\vec{v}^T \vec{v}}_{\text{dot product}}}$

$$\Rightarrow \underbrace{\|A\vec{x}\| = \sqrt{(A\vec{x})^T A\vec{x}} = \sqrt{\vec{x}^T A^T A \vec{x}} = \sqrt{\vec{x}^T \vec{x}} = \|\vec{x}\|}_{\text{pf for } n\times n \text{ case}}$$

⚠ A has to be orthogonal!

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1. False: let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)^2 = 0;\ \lambda = 1.$

$\text{Nul}(A - 1I) : \text{Nul}\left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ x_1 free, x_2 free

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; e-vectors $= \Big\{ \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\text{lin ind; same e-value}} \Big\}$

2. True: This follows from the def of e-value/ e-vector: $(A-\lambda I)\vec{x} = \vec{0} \iff A\vec{x} - \lambda \underbrace{I\vec{x}}_{\vec{x}} = \vec{0}$

$\iff A\vec{x} = \lambda\vec{x} \iff \lambda$ is an e-value of A associated with e-vector $\vec{x} \neq \vec{0}$

3. False: Counterexample

$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \rightarrow$ e-values $\lambda = 2, 2$

$\wr$ REF

$\begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix} \rightarrow$ e-values $\lambda = 1, 4$

5. Two matrices that have the same eigenvalues (with same multiplicities) should be similar.

6. If A (square matrix) is diagonalizable and invertible, then (choose the right answer)

A. Its inverse is diagonalizable and is invertible

B. Its inverse is diagonalizable and could be invertible

C. Its inverse could be diagonalizable and is invertible

D. Its inverse could be diagonalizable and could be invertible

E. We need more information to determine this

5. $\underbrace{A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{\lambda_A = 1 \text{ (multip. 2)}}$ $\underbrace{B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\lambda_B = 1 \text{ (multi. 2)}}$

$A = MBM^{-1} \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = M \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} M^{-1}$

$= MM^{-1} \neq I$

6. If A (square matrix) is diagonalizable and invertible, then (choose the right answer)

A. Its inverse is diagonalizable and is invertible ✓

B. ~~Its inverse is diagonalizable and could be invertible~~

C. ~~Its inverse could be diagonalizable and is invertible~~

D. ~~Its inverse could be diagonalizable and could be invertible~~

E. ~~We need more information to determine this~~

$(A^{-1})^{-1} = A$

A d'ble: $\exists M \mid A = M\Lambda M^{-1}$

A invertible: A^{-1} exists

$A^{-1} = (M\Lambda M^{-1})^{-1} = (M^{-1})^{-1}\Lambda^{-1}M^{-1}$

$\boxed{A^{-1} = M\Lambda^{-1}M^{-1}}$