Linear Algebra

Final Sample Question

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Exercise True or False? In both cases, explain clearly.

- $\det(AB) = \det(BA)$ Yes
- Q's column vectors are orthogonal, $QQ^\top$ is a projection matrix. No
- For unit vector $\|u\| = 1$, $Q = I - 2uu^\top$ is an orthogonal matrix

  (This is called Householder Matrix) Yes! first Q is symmetric because $uu^\top$ is symmetric. Then $QQ^\top = QQ = Q^2 = (I - 2uu^\top)^2 = I - 2uu^\top - 2uu^\top + 4u \underbrace{u^\top u}_{u^\top u = \|u\|^2 = 1} u^\top = I - 4uu^\top + 4uu^\top = I$
- Q's column vectors are orthonomral, $QQ^\top$ is a projection matrix. Yes
- A^10 is invertible then A^3 is also invertible. Yes
- $\text{rank}(A) = \text{rank}(A^\top A)$ Yes
- Suppose you remove row 3 and column 4 of an invertible 5 by 5 matrix A. If that reduced matrix is not invertible, then the $(4, 3)-$ entry of A^{-1} is 0. Yes
- Suppose you remove row 3 and column 4 of an invertible 5 by 5 matrix A. If that reduced matrix is not invertible, then the $(3, 4)-$ entry of A^{-1} is 0. No
- If $I - A^2$ is invertible, then $I - A$ is invertible Yes, consider eigen value of A can't be $1, -1$.
- If A is a real $n \times n$ matrix then $\det(A^2) \geq 0$.

  Yes
- If A is an $n \times n$ matrix with n distinct real eigenvalues, then A is diagonalizable.

  Yes

- If A is a positive definite, then $A^\top + I$ is also positive definite matrix. Yes
- All positive definite matrices invertible

  Yes

Hard Questions

- For a 3×3 matrix A, if $\det(A) = 3$, what is $\det(C)$ where C is the matrix of cofactor?
- What are the four fundamental subspaces of $M = I - P$ in terms of the column space of P

  Left Null space = Right Null space = Colume space of P. Column space = Row space = orthogonal complement of the colume space of P.

  This means that $P_{\text{row}(A)} + P_{\text{Nul}(A)} = I$, projection to the column space+ projection to the null space is identity.

- If P is projection matrix, then
  - Check $P^\top = P$
  - $P^2 = P$ (means $\det(P) = 1$ or 0)

$$P^2 = \underbrace{A(A^\top A)^{-1}A^\top}_{P} A(A^\top A)^{-1}A^\top = A\underbrace{(A^\top A)^{-1}A^\top A}_{I}(A^\top A)^{-1}A^\top = A(A^\top A)^{-1}A^\top = P$$

  - $I - 2P$ is an orthogonal matrix

    Since P is a projection matrix, we have $P = P^T$. To show that Q is an orthogonal matrix, we need to check that $QQ^T = I$. We have

$$QQ^T = (I - 2P)(I - 2P)^T$$

$$= (I - 2P)(I^T - 2P^T)$$

$$= (I - 2P)(I - 2P) \quad \text{(since } I \text{ and } P \text{ are symmetric)}$$

$$= I - 4P + 4P^2$$

    Since for a projection matrix we have $P^2 = P$, this product is equal to $QQ^T = I$, as required.
  - If P is full rank, then $P = I$.

    If P is full rank means A in $P = A(A^\top A)^{-1}A^\top$ (A and P has the full column space). Then

$$P = A(A^\top A)^{-1}A^\top = AA^{-1}(A^\top)^{-1}A^\top = II = I$$

- Suppose an $m \times n$ matrix A has rank r. What are the ranks of

  1. A^T?
  2. AA^T?
  3. $AA^T + \lambda I$ $(\lambda > 0)$?
  4. $A^T AA^T$?

  Solution

  Answer 1

  (A) r

  (B) we showed in class it's r (page 17 in `https://2prime.github.io/files/linear/linearslide14filled.pdf`)

  (C) it's a positive definite matrix with all eigenvalues lareger than r, think why.

  (D) r (similar page 17 in `https://2prime.github.io/files/linear/linearslide14filled.pdf`)

  Answer 2 Using SVD

  (A) $\text{rank}(A^T) = \dim(\text{row}(A^T)) = \dim(\text{col}(A)) = \text{rank}(A) = r$.

  (B) Let $A = U\Sigma V^T$ be a full SVD. Then,

  $$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = U\Sigma V^T V\Sigma^T U^T = U\Sigma^2 U^T.$$

  Thus, $U\Sigma^2 U^T$ is a SVD of AA^T. If Σ has r positive singular values then so will Σ^2. Therefore, the rank of AA^T is r.

  (C) Since $I_m = UU^T$, the equation above yields $AA^T + \lambda I = U\Sigma^2 U^T + \lambda I = U(\Sigma^2 + \lambda I_m)U^T$. Since $\Sigma^2 + \lambda I = \text{diag}(\sigma_1^2 + \lambda, \ldots, \sigma_r^2 + \lambda, \ldots, \lambda)$, the rank is m.

  (D) $A^T AA^T = (U\Sigma V^T)^T (U\Sigma V^T)(U\Sigma V^T)^T = V\Sigma^T U^T U\Sigma V^T U^T U\Sigma V^T = V\Sigma^T \Sigma\Sigma^T V^T = V\Sigma^4 V^T$. Σ^2 has r positive singular values as like Σ. Therefore, the rank is r.

Questions

- Compute Projections, Learst square solution, QR Decomposition
- Compute Determinate using Cofactor Expansion, Solve linear system using cramer's rule
- Compute Eigenvalues, compute eigenvectors, $A = XBX^{-1}$ how to calculate X

Problem 1

Find the determinant of $I + M$, if M is the rank one matrix $M = vv^T$, where v is a column vector $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$$I + M = \begin{bmatrix} 1+a^2 & ab & ac \\ ab & 1+b^2 & bc \\ ac & bc & 1+c^2 \end{bmatrix}$$

$$\begin{aligned}
\det(I+M) &= \det \begin{bmatrix} 1+a^2 & ab & ac \\ ab & 1+b^2 & bc \\ ac & bc & 1+c^2 \end{bmatrix} \\
&= (1+a^2)\det\begin{bmatrix} 1+b^2 & bc \\ bc & 1+c^2 \end{bmatrix} - ab\det\begin{bmatrix} ab & bc \\ ac & 1+c^2 \end{bmatrix} + ac\det\begin{bmatrix} ab & 1+b^2 \\ ac & bc \end{bmatrix} \\
&= (1+a^2)((1+b^2)(1+c^2) - bc\cdot bc) - ab(ab(1+c^2) - ac\cdot bc) + ac(ab\cdot bc - (1+b^2)ac) \\
&= 1 + a^2 + b^2 + c^2
\end{aligned}$$

If you know Sylvester's determinant theorem, the problem becomes incredibly easy. Applied to this particular instance, we get that $\det(I + vv^\top) = \det(1 + v^\top v) = 1 + \|v\|^2$

Problem 2

The following matrices have only one eigenvalue: 1. What are the dimensions of the eigenspaces in each case?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

For a matrix A, the eigenspace with eigenvalue λ is the kernel of the matrix $A - \lambda I$. Here we have $\lambda = 1$, so we subtract I from each of the matrices above:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and find the dimensions of the kernels.

The ranks of these matrices are 0, 2, 2, 1 respectively, so by the rank-nullity theorem the dimensions of the kernels are 3, 1, 1, 2.

Answer: 3, 1, 1, 2.

Problem 3

Every permutation matrix leaves $x = (1, 1, \ldots, 1)$ unchanged. Then $\lambda = 1$. Find two more λ's (possibly complex) for these permutations P_1 and P_2, from $\det(P - \lambda I) = 0$:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Solution:

- $\det(P_1 - \lambda I) = 1 - \lambda^3$, therefore $\lambda_1 = 1$, $\lambda_2 = \frac{-1+i\sqrt{3}}{2}$, $\lambda_3 = \frac{-1-i\sqrt{3}}{2}$.
- $\det(P_2 - \lambda I) = (\lambda - 1)(\lambda^2 + \lambda - 1)$, therefore $\lambda_1 = -1$, $\lambda_2 = \lambda_3 = 1$.

Problem 4

Diagonalize A and compute $V\mathbf{A}kV^{-1}$ to prove this formula for A^k:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \text{ has } A^k = \frac{1}{2}\begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix}.$$

Solution:

The eigenvalues of A are 3 and 1, and the corresponding eigenvectors are $v_1 = (-1, 1)$, $v_2 = (1, 1)$. Therefore, A can be diagonalized as $A = VAV^{-1}$, where $V = [v_1, v_2]$, $\Lambda = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ and $V^{-1} = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. $A^k = V\Lambda^k V^{-1} = \frac{1}{2}\begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix}$.

Problem 5

Without multiplying

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

find:

(a) the determinant of A,

(b) the eigenvalues of A,

(c) the eigenvectors of A,

(d) the reason why A is symmetric positive definite.

Solution:

What we have to notice is that we are given a $Q\Lambda Q^{-1} = Q\Lambda Q^T$ decomposition of the matrix, so all the information that we want can be read off from it. For (a) the determinant is equal to the determinant of the diagonal matrix $2 \cdot 5 = 10$. For (b), we have that the eigenvalues are 2 and 5. For (c), the eigenvectors are the columns of Q, so $[\cos\theta \quad \sin\theta]^T$ and $[-\sin\theta \quad \cos\theta]^T$. The matrix is clearly symmetric since $(Q\Lambda Q^T)^T = Q\Lambda Q^T$ and its eigenvalues are positive, so it is positive-definite.