Properties of Determinate

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Four Basic Property

- (1) $\det([v_1, \cdots, v_i, \cdots, v_j, \cdots, v_n]) = -\det([v_1, \cdots, v_j, \cdots, v_i, \cdots, v_n])$ (switching i, j column make the determinate negative)
- (2) $\det(AB) = \det(A)\det(B)$
- (linear combination of single column)
  - (3) $\det([v_1 + v_1', v_2, \cdots, v_n]) = \det([v_1, v_2, \cdots, v_n]) + \det([v_1', v_2, \cdots, v_n])$
  - (4) $\det([cv_1, v_2, \cdots, v_n]) = c\det([v_1, v_2, \cdots, v_n])$

determinates of basic matrix

- $\det(I_n) = I$
- What is the determinate of a diagonal matrix? $\det\left[\begin{pmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{pmatrix}\right] = a_{11} a_{22} a_{33}$
- What is the determinate of an orthogonal matrix matrix?
- What is the determinate of a permutation matrix?
- What is the determinate of an elimination matrix? 1
- What is the determinate of a lower diagonal matrix? $\det\left[\begin{pmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right] = a_{11} a_{22} a_{33}$

$= \det\left[\begin{pmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ & & a_{33} \end{pmatrix}\right] = \det\left[\begin{pmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{pmatrix}\right]$

Try to prove the following property:

- $\det([\vec{0}, v_1, v_2, \cdots, v_{n-1}]) = 0$
- $\det([v_1, v_1, v_2, \cdots, v_{n-1}]) = 0$ (A have two equal columns, then determinante is 0)
- $\det(cA) = c^n \det(A)$

  all things in the note generalize to row operations!

  2×2: $\det(cA) = \det([cv_1 . cv_2]) = c\det([v_1 . cv_2])$
  $= c \cdot c \det([v_1 . v_2])$
- $\det(A) = \det(A^T)$ (*hint*:LU Decomposition)

$A = L \cdot U$

$\det(A) = \det(L) \cdot \det(U)$

$l_{11} \cdots l_{nn}$ $\quad u_{11} \cdots u_{nn}$

$A^T = U^T L^T$

$\det(A^T) = \det(U^T) \det(L^T)$

$u_{11} \cdots u_{nn}$ $\quad l_{11} \cdots l_{nn}$

Example. $\det([v_1, -v_1, v_2]) = 0$

$\Downarrow$

(1) $\det([v_1, -v_1, v_2]) = -\det([-v_1, v_1, v_2])$ switching two columns

(2) $\det([v_1, -v_1, v_2]) \overset{(-1)\times(-1)}{=} \det([-v_1, v_1, v_2])$ (?)

The first column times (−1)
The second column times (−1)

By (1) and (2) $\det([v_1, -v_1, v_2]) = 0$

Col. $\det([v_1, v_1, v_2]) = 0$

I right multiply E_{21}, but not left
column operation — row operation

$[v_1, v_2]E_{21}$

Example. $\det([v_1 + jv_2, v_2]) = \det([v_1, v_2])$

$$\det([v_1, v_2]) + j\underbrace{\det([v_2, v_2])}_{=0} = \det([v_1, v_2])$$

Remark $\det(E_{ij}) = 1$ E_{ij} elimination matrix