Dimensionality Reduction

Using linear algebra

Motivation

- Clustering
  - One way to summarize a complex real-valued data point with a single categorical variable
- Dimensionality reduction
  - Another way to simplify complex high-dimensional data
  - Summarize data with a lower dimensional real valued vector

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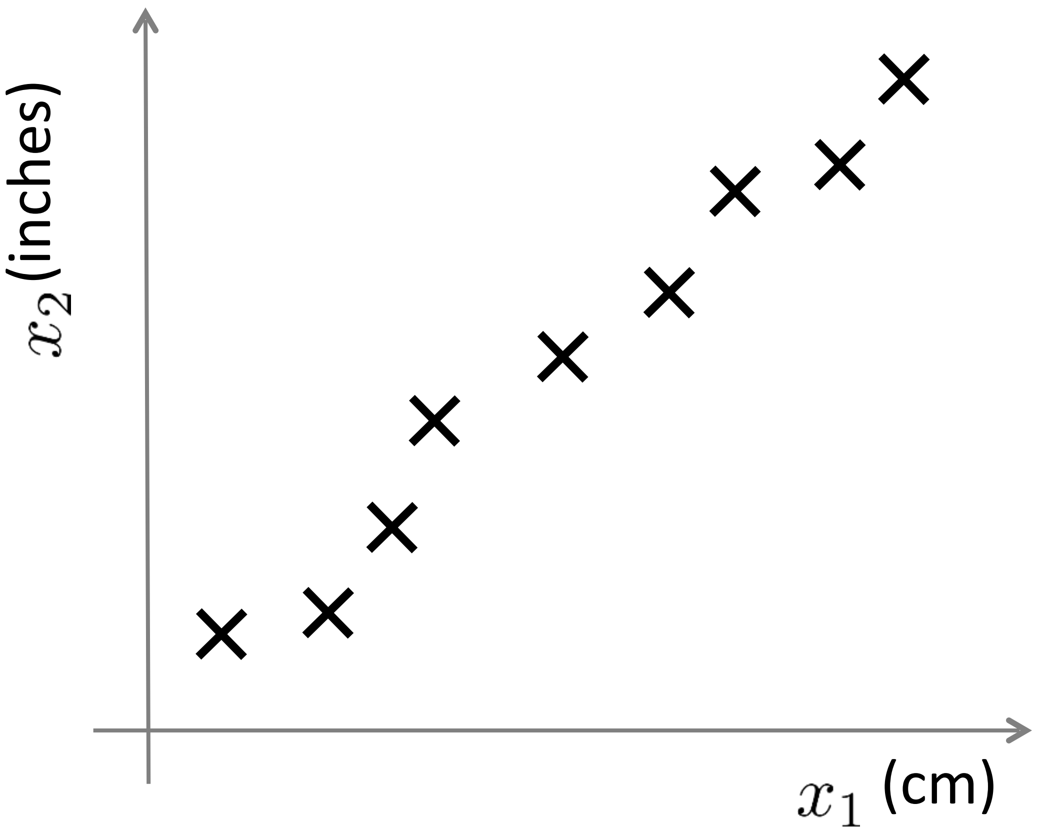
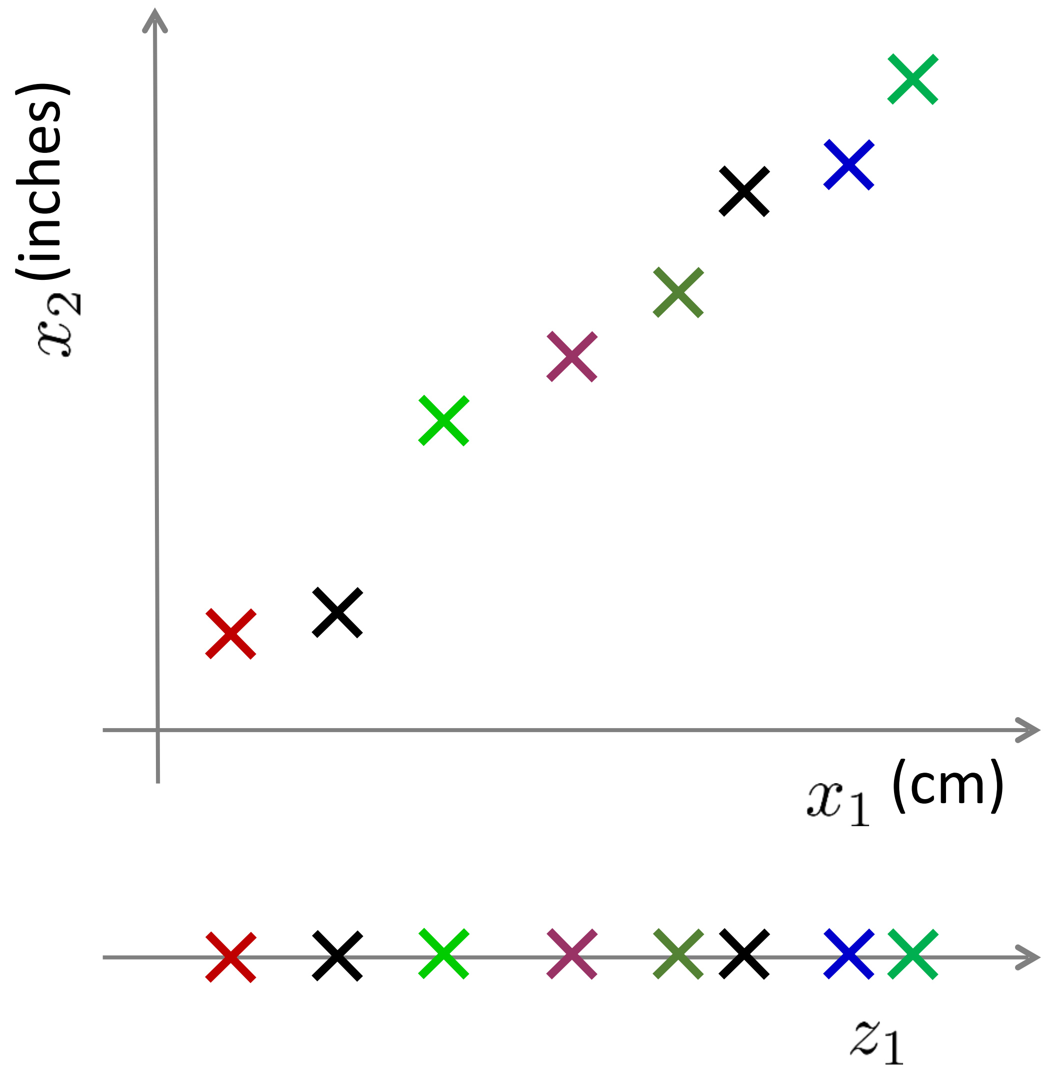
- Given data points in d dimensions
- Convert them to data points in $r<d$ dimensions
- With minimal loss of information

Data Compression



Reduce data from 2D to 1D

Data Compression

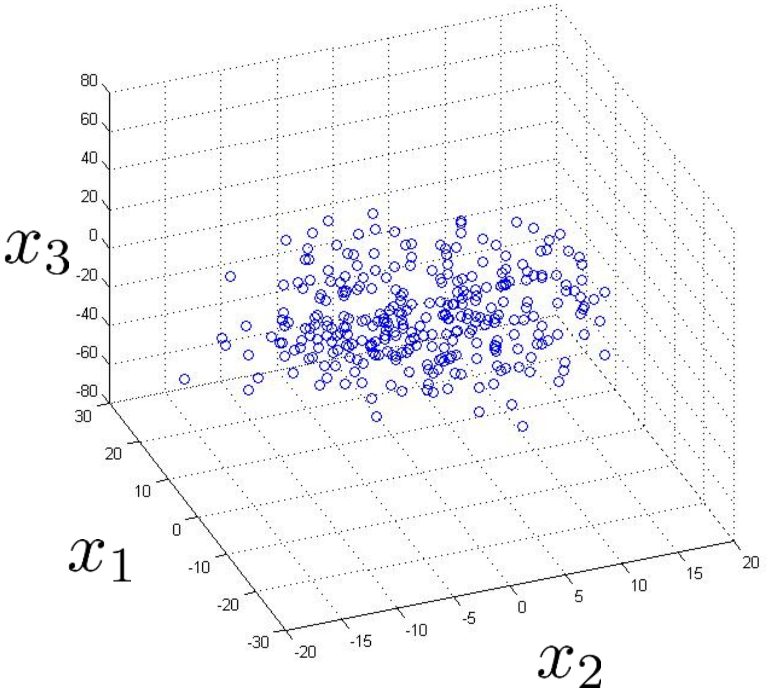


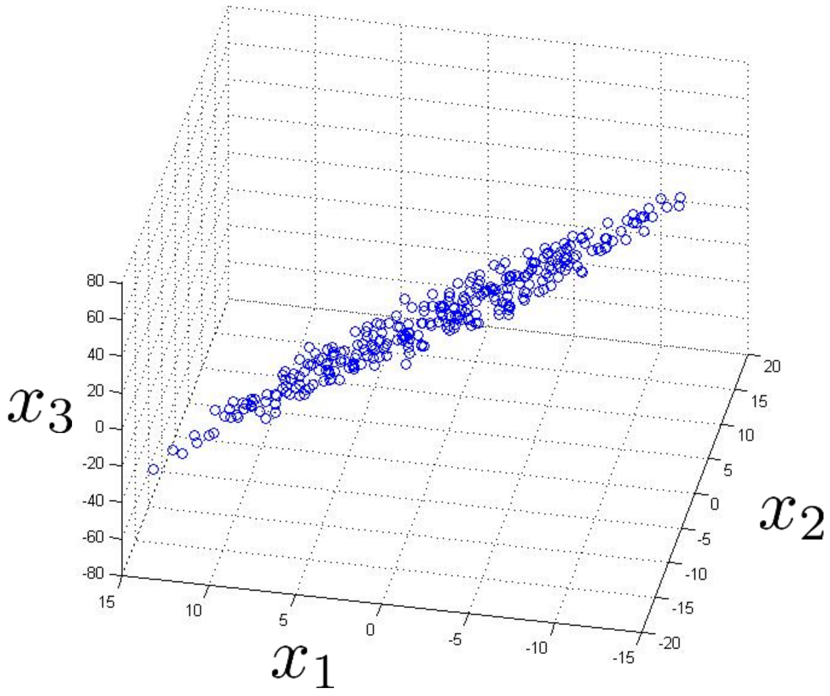
Reduce data from 2D to 1D

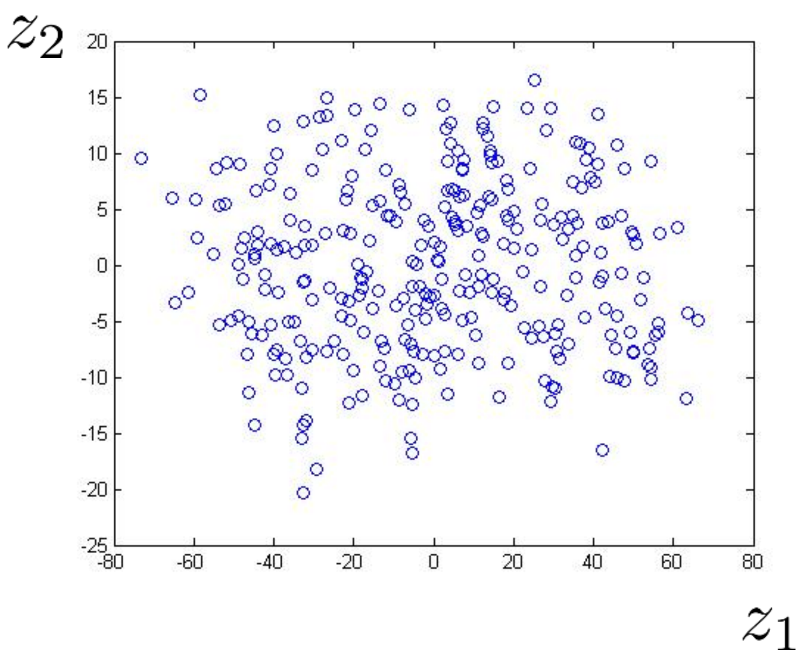
$$x^{(1)} \rightarrow z^{(1)}$$
$$x^{(2)} \rightarrow z^{(2)}$$
$$\vdots$$
$$x^{(m)} \rightarrow z^{(m)}$$

Data Compression

Reduce data from 3D to 2D

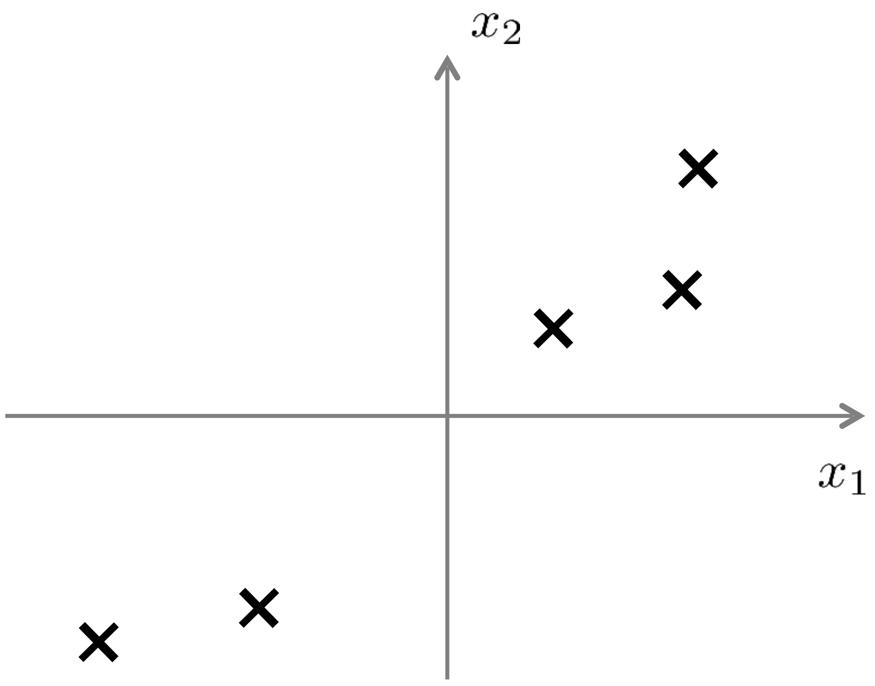


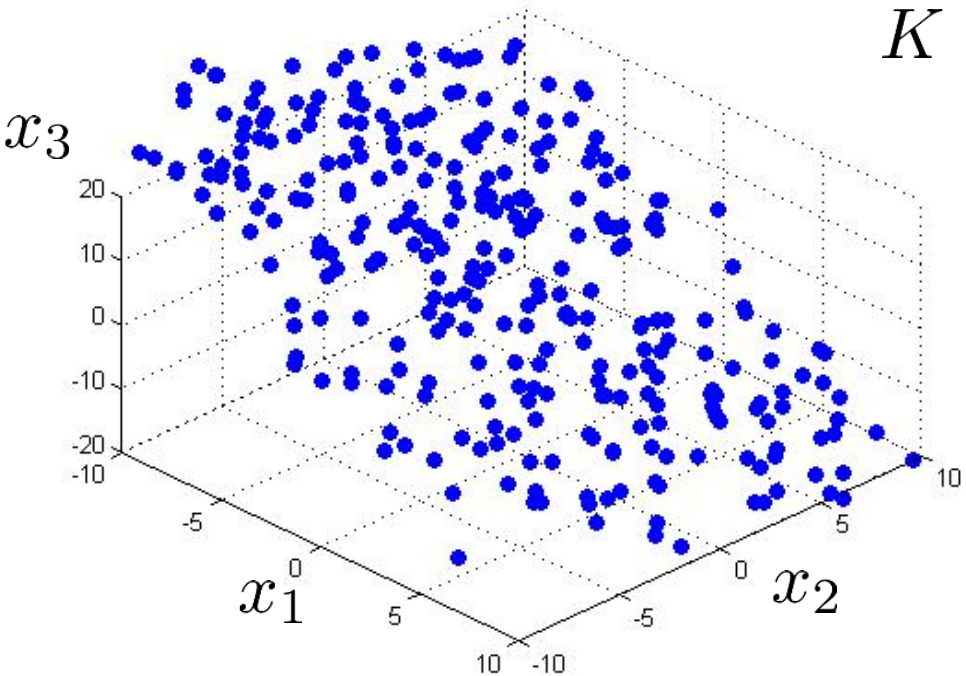




Principal Component Analysis (PCA) problem formulation

$3D \rightarrow 2D$
$K = 2$





Reduce from 2-dimension to 1-dimension: Find a direction (a vector $u^{(1)} \in \mathbb{R}^n$) onto which to project the data so as to minimize the projection error.

Reduce from n-dimension to k-dimension: Find k vectors $u^{(1)}, u^{(2)}, \dots, u^{(k)}$ onto which to project the data, so as to minimize the projection error.

Principal Component Analysis

Goal: Find r-dim projection that best preserves variance

1. Compute mean vector μ and covariance matrix Σ of original points
2. Compute eigenvectors and eigenvalues of Σ
3. Select top r eigenvectors
4. Project points onto subspace spanned by them:

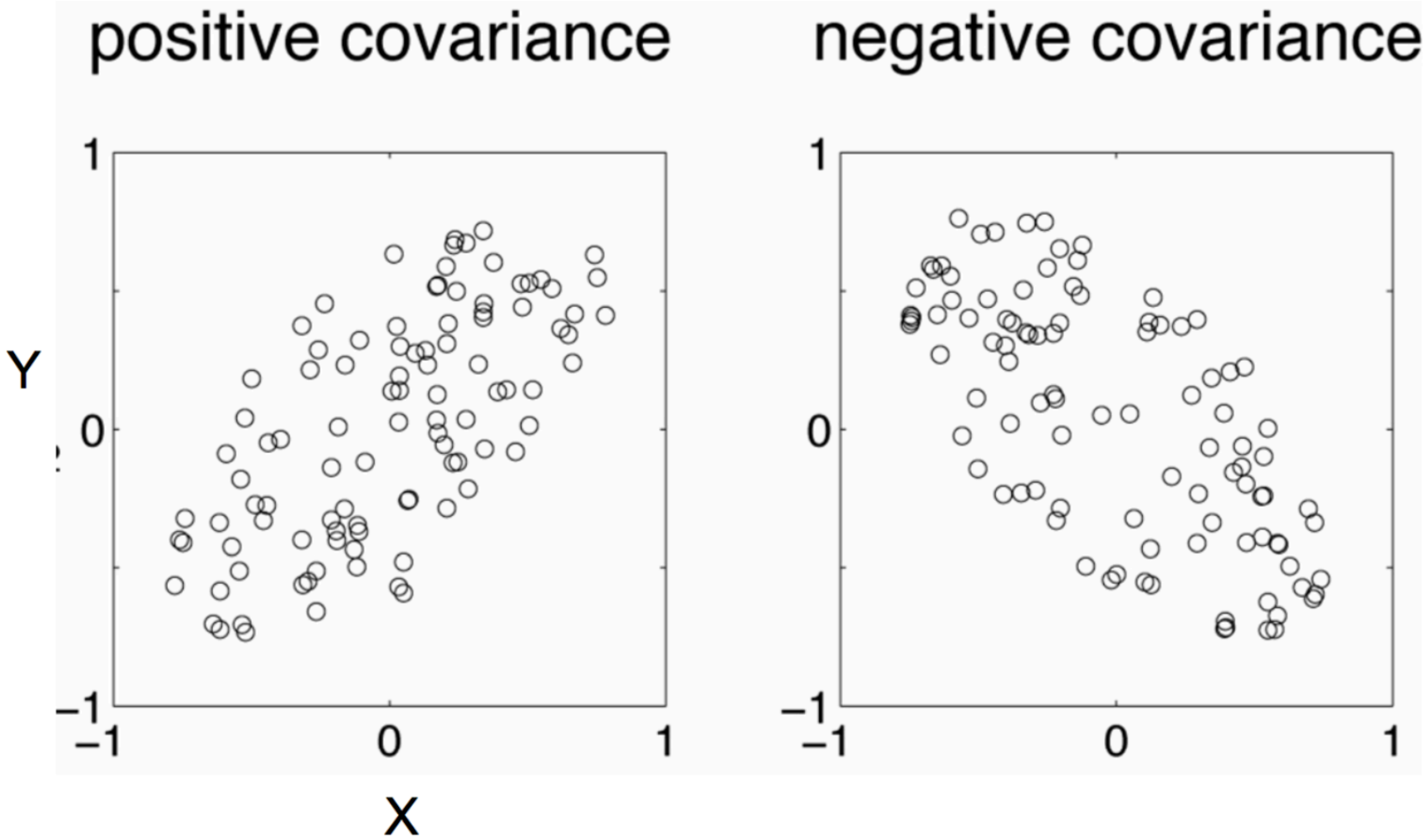
$$y = A(x - \mu)$$

   where y is the new point, x is the old one, and the rows of A are the eigenvectors

Covariance

- Variance and Covariance:
  - Measure of the “spread” of a set of points around their center of mass(mean)
- Variance:
  - Measure of the deviation from the mean for points in one dimension
- Covariance:
  - Measure of how much each of the dimensions vary from the mean with **respect to each other**

- **Covariance is measured between two dimensions**
- **Covariance sees if there is a relation between two dimensions**
- **Covariance between one dimension is the variance**



Positive: Both dimensions increase or decrease together

Negative: While one increase the other decrease

Covariance

- Used to find relationships between dimensions in high dimensional data sets

$$q_{jk} = \frac{1}{N} \sum_{i=1}^{N} (X_{ij} - E(X_j))(X_{ik} - E(X_k))$$

The Sample mean

Eigenvector and Eigenvalue

$$Ax = \lambda x$$

A: Square Matirx

λ: Eigenvector or characteristic vector

X: Eigenvalue or characteristic value

- ***The zero vector can not be an eigenvector***
- ***The value zero can be eigenvalue***

Eigenvector and Eigenvalue

Ax = λx

A: Square Matirx

λ: Eigenvector or characteristic vector

X: Eigenvalue or characteristic value

Example

$$Show\ x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} is\,an\,eigenvector\ for\ A = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}$$

$$Solution: Ax = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$But\ \ for\ \ \lambda = 0,\ \ \lambda x = 0\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$Thus, x\,is\,an\,eigenvector\,of\ A, and\,\lambda = 0\ is\,an\,eigenvalue.$$

Eigenvector and Eigenvalue

$Ax = \lambda x$ → $Ax - \lambda x = 0$

$(A - \lambda I)x = 0$

If we define a new matrix B: → $B = A - \lambda I$

$Bx = 0$

If B has an inverse: → $x = B^{-1}0 = 0$ ✗ BUT! an eigenvector cannot be zero!!

x will be an eigenvector of A if and only if B does not have an inverse, or equivalently det(B)=0 :

$$\det(A - \lambda I) = 0$$

Eigenvector and Eigenvalue

Example 1: Find the eigenvalues of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & 12 \\ -1 & \lambda + 5 \end{vmatrix} = (\lambda - 2)(\lambda + 5) + 12$$

$$= \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2)$$

two eigenvalues: –1, – 2

Note: The roots of the characteristic equation can be repeated. That is, $\lambda_1 = \lambda_2 = \ldots = \lambda_k$. If that happens, the eigenvalue is said to be of multiplicity k.

Example 2: Find the eigenvalues of

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

$\lambda = 2$ is an eigenvector of multiplicity 3.

Principal Component Analysis

Input: $\mathbf{x} \in \mathbb{R}^D: \ \mathcal{D} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$

Set of basis vectors: $\mathbf{u}_1, \ldots, \mathbf{u}_K$

Summarize a *D* dimensional vector *X* with *K* dimensional feature vector *h(x)*

$$h(\mathbf{x}) = \begin{bmatrix} \mathbf{u}_1 \cdot \mathbf{x} \\ \mathbf{u}_2 \cdot \mathbf{x} \\ \cdots \\ \mathbf{u}_K \cdot \mathbf{x} \end{bmatrix}$$

Principal Component Analysis

$$\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_K]$$

Basis vectors are orthonormal

$$\mathbf{u}_i^T \mathbf{u}_j = 0$$
$$||\mathbf{u}_j|| = 1$$

New data representation *h(x)*

$$z_j = \mathbf{u}_j \cdot \mathbf{x}$$
$$h(\mathbf{x}) = [z_1, \ldots, z_K]^T$$

Principal Component Analysis

$$\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_K]$$

New data representation *h(x)*

$$h(\mathbf{x}) = \mathbf{U}^T \mathbf{x}$$

$$h(\mathbf{x}) = \mathbf{U}^T (\mathbf{x} - \mu_0)$$

Empirical mean of the data →

$$\mu_0 = \frac{1}{N} \sum_{i=1}^{N} \mathbf{x}_i$$

The space of all face images

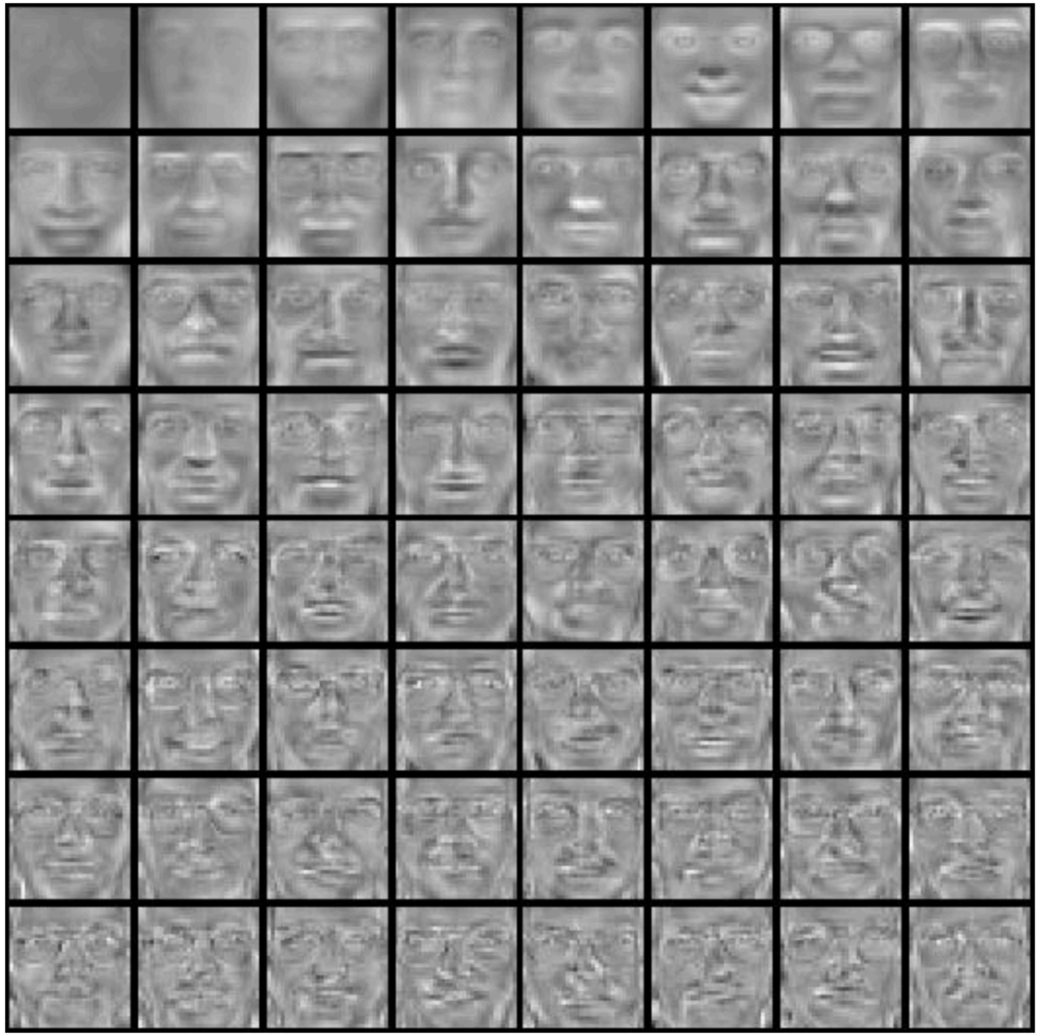
- When viewed as vectors of pixel values, face images are extremely high-dimensional
  - 100x100 image = 10,000 dimensions
  - Slow and lots of storage
- But very few 10,000-dimensional vectors are valid face images
- We want to effectively model the subspace of face images

slide by Derek Hoiem

Eigenfaces example

Mean: μ

Top eigenvectors: $u_1, \ldots u_k$



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Representation and reconstruction

- Face **x** in “face space” coordinates:

$$\mathbf{x} \rightarrow [\mathbf{u}_1^{\mathrm{T}}(\mathbf{x}-\mu), \ldots, \mathbf{u}_k^{\mathrm{T}}(\mathbf{x}-\mu)]$$
$$= \quad w_1, \ldots, w_k$$

- Reconstruction:

 =  + 

$$\hat{x} = \mu + w_1 u_1 + w_2 u_2 + w_3 u_3 + w_4 u_4 + \ldots$$

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Reconstruction

P = 4

P = 200

P = 400

After computing eigenfaces using 400 face images from ORL face database

Application: Image compression

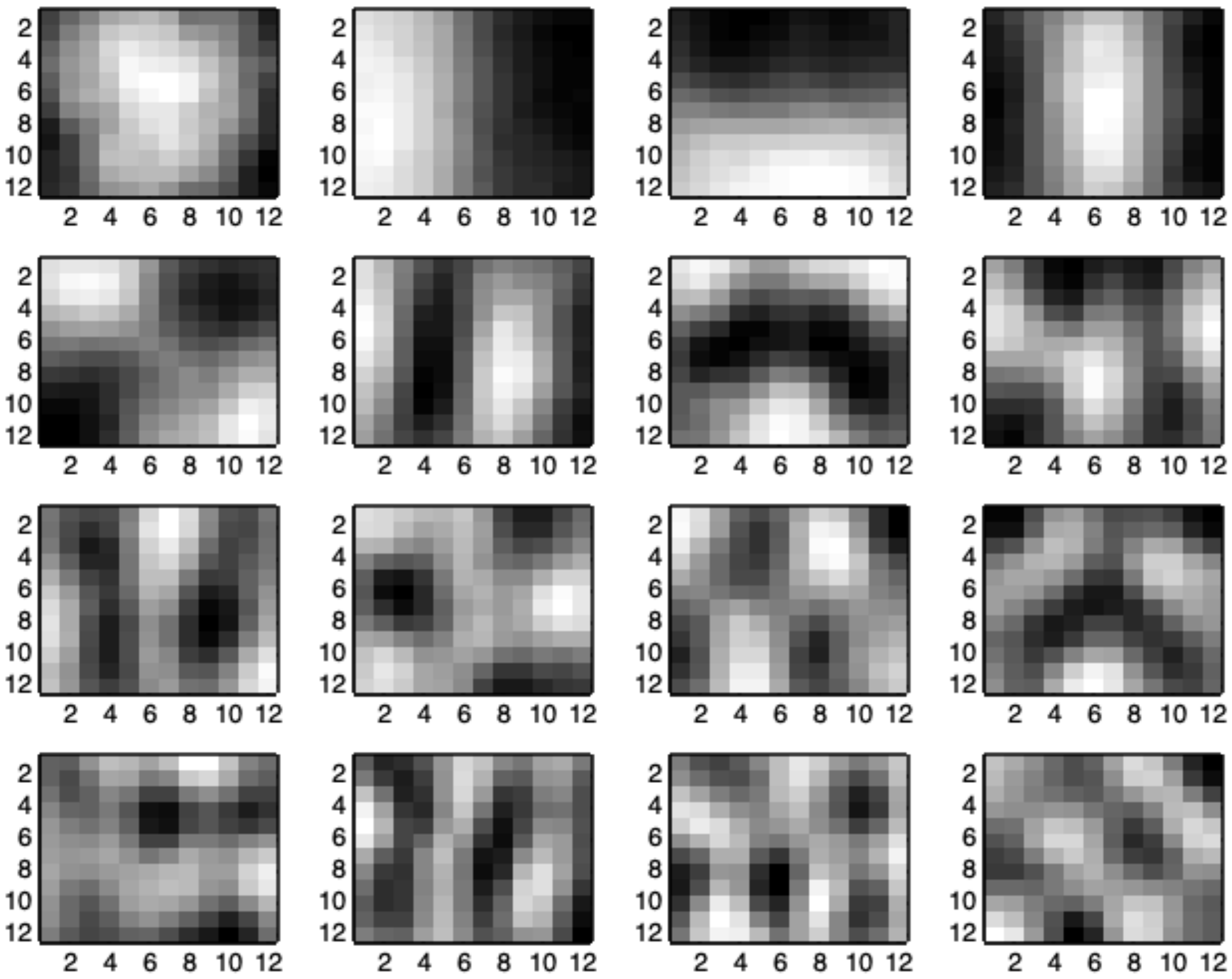
Original Image

- Divide the original 372x492 image into patches:
  - Each patch is an instance that contains 12x12 pixels on a grid
- View each as a 144-D vector

PCA compression: 144D → 60D

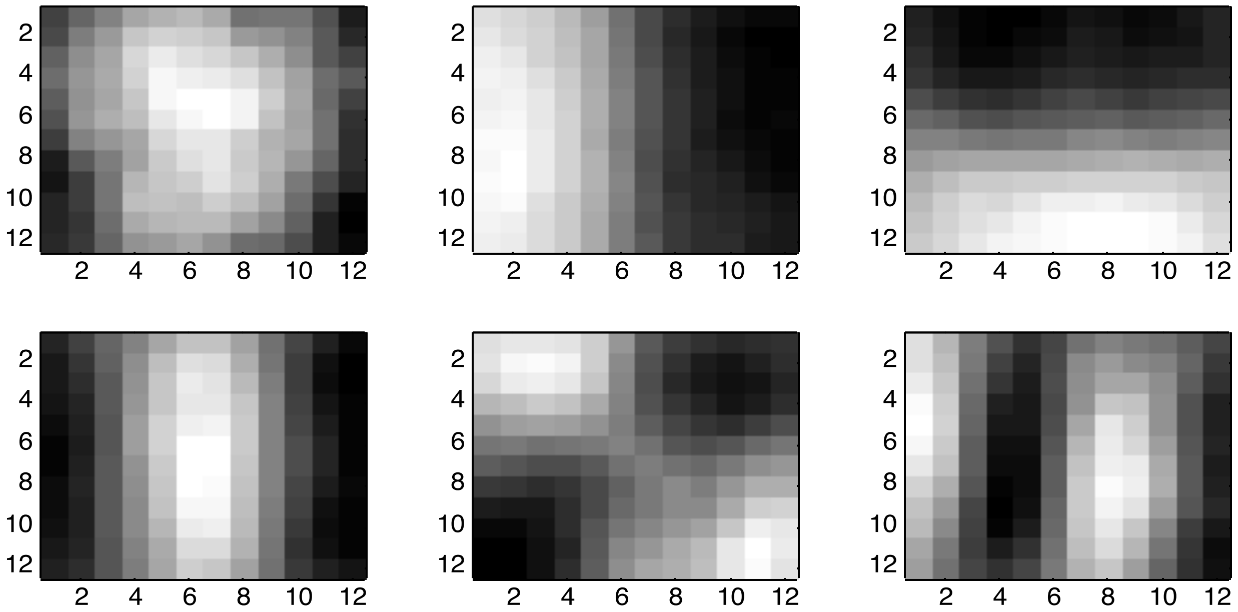
PCA compression: 144D → 16D

16 most important eigenvectors



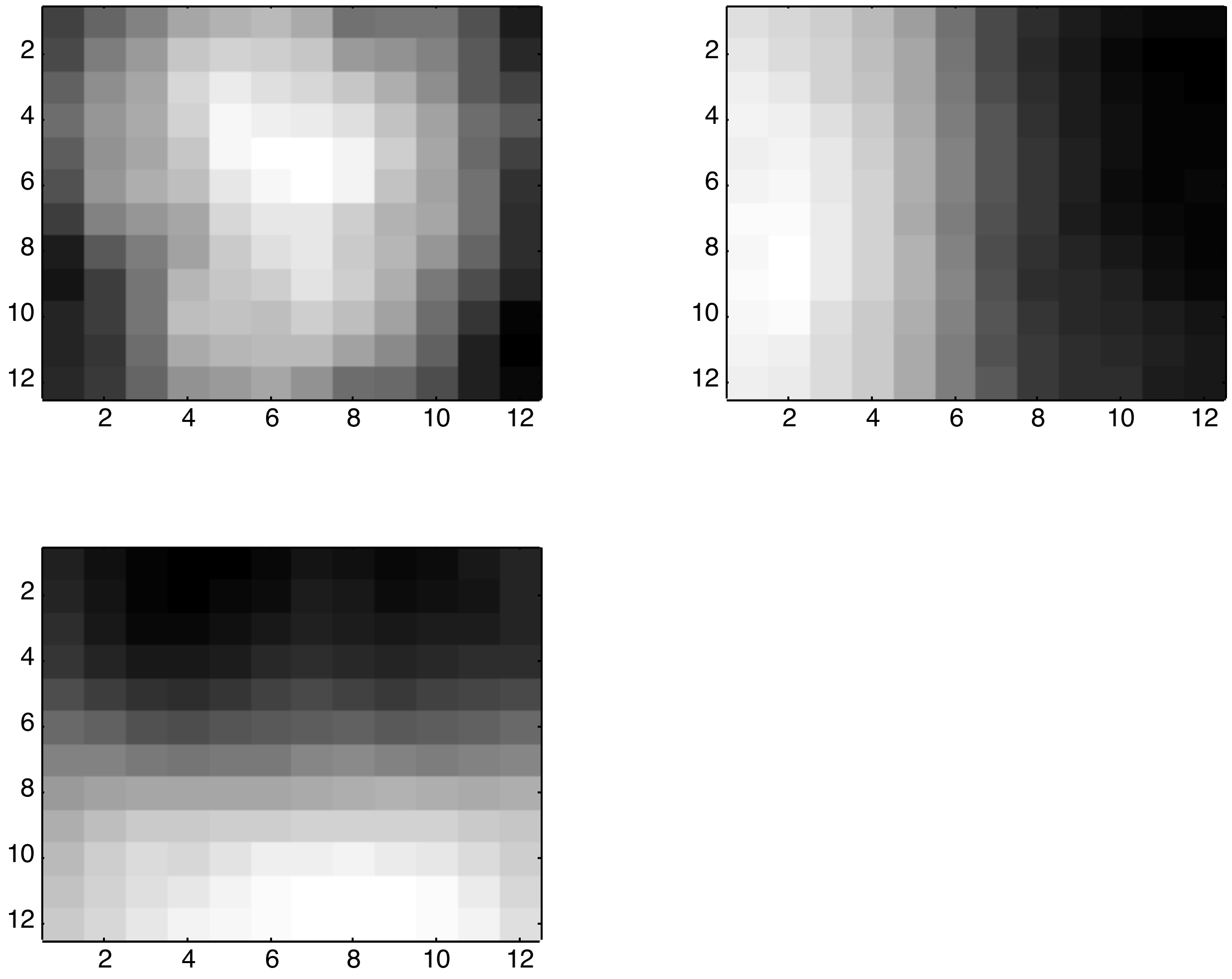
PCA compression: 144D) 6D

6 most important eigenvectors

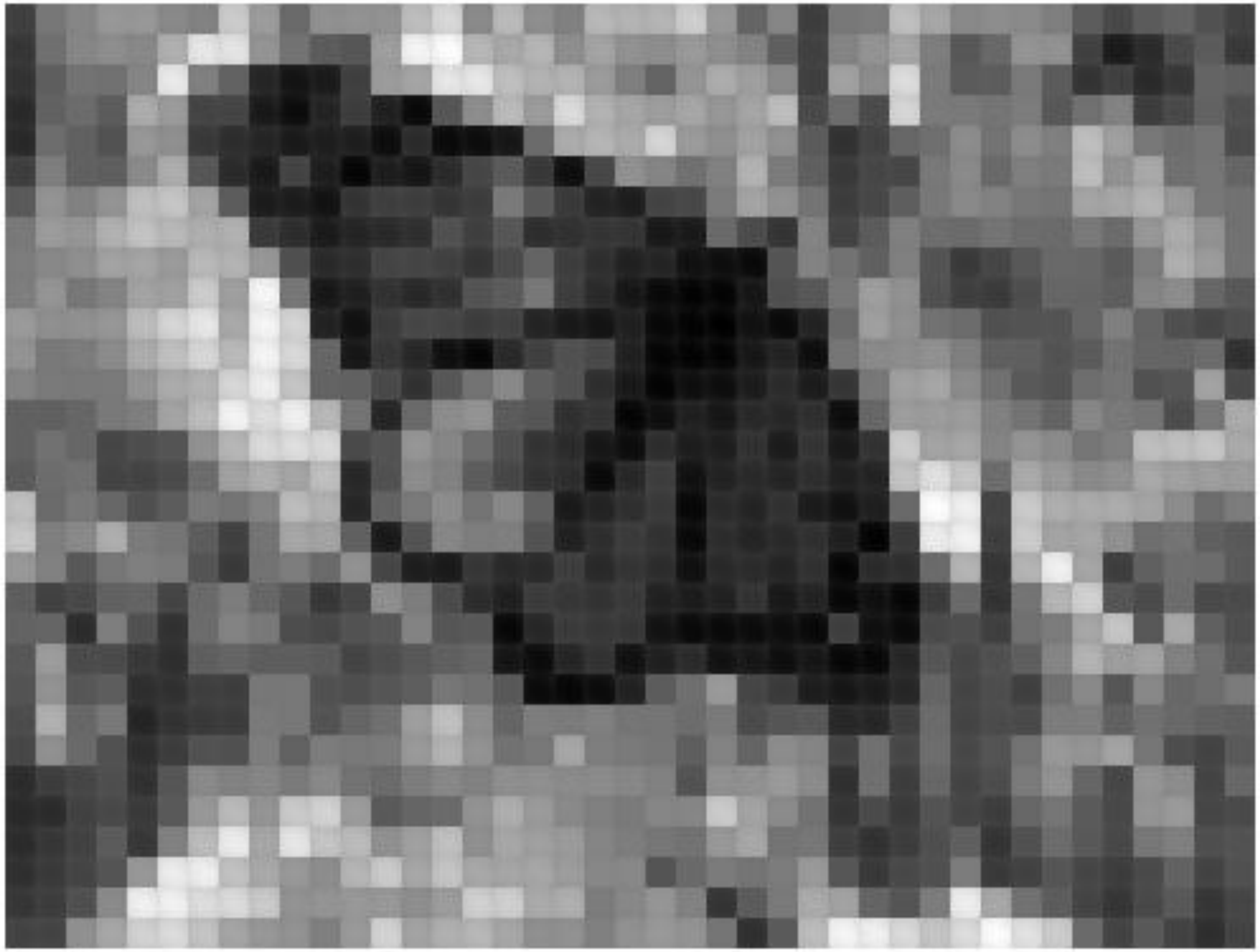


PCA compression: 144D → 3D

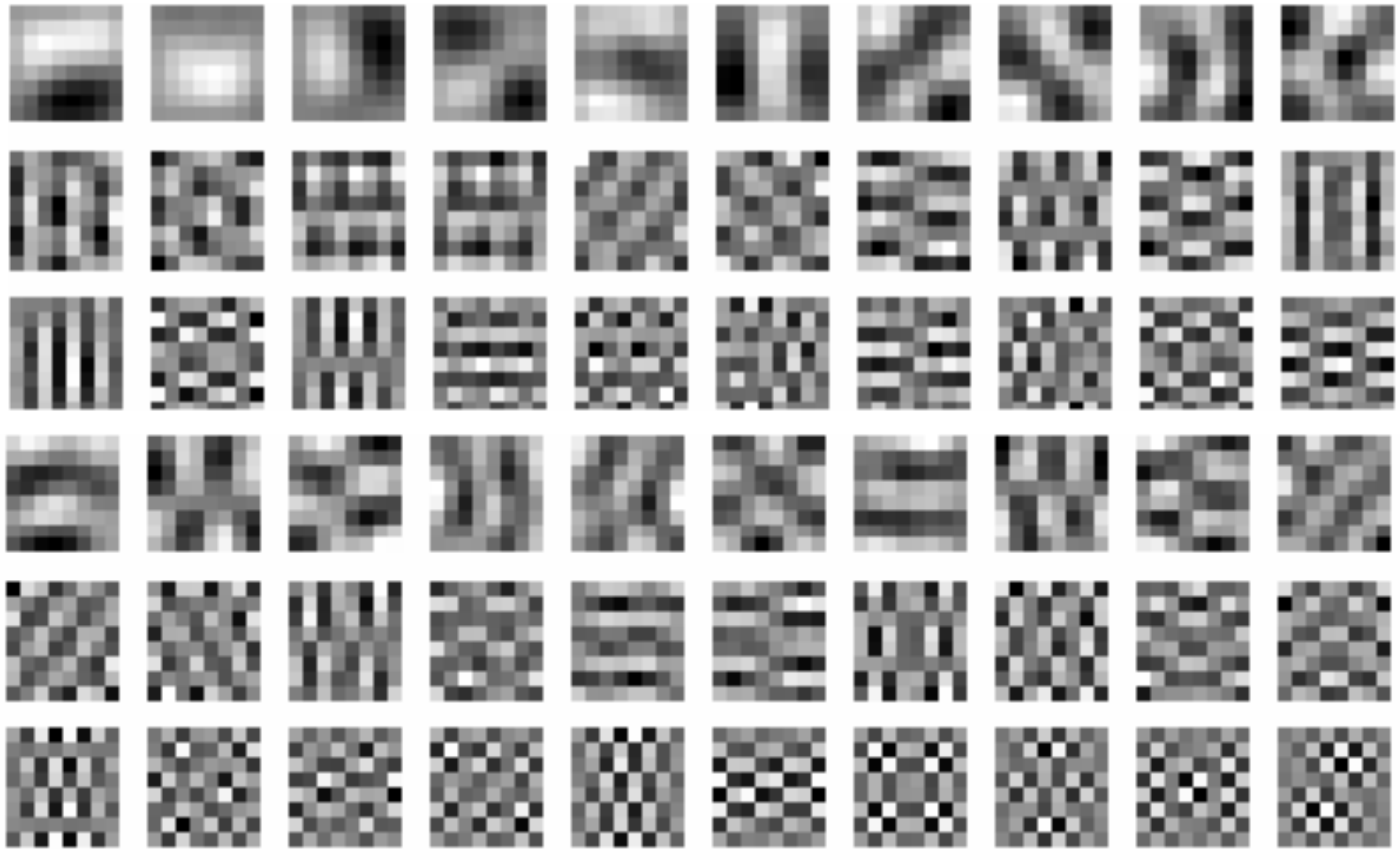
3 most important eigenvectors



PCA compression: 144D → 1D

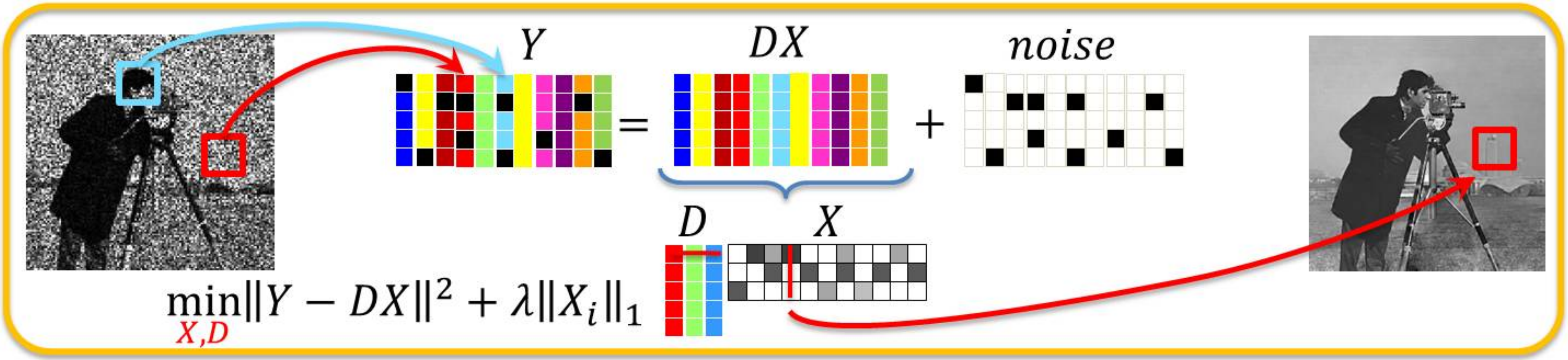


60 most important eigenvectors

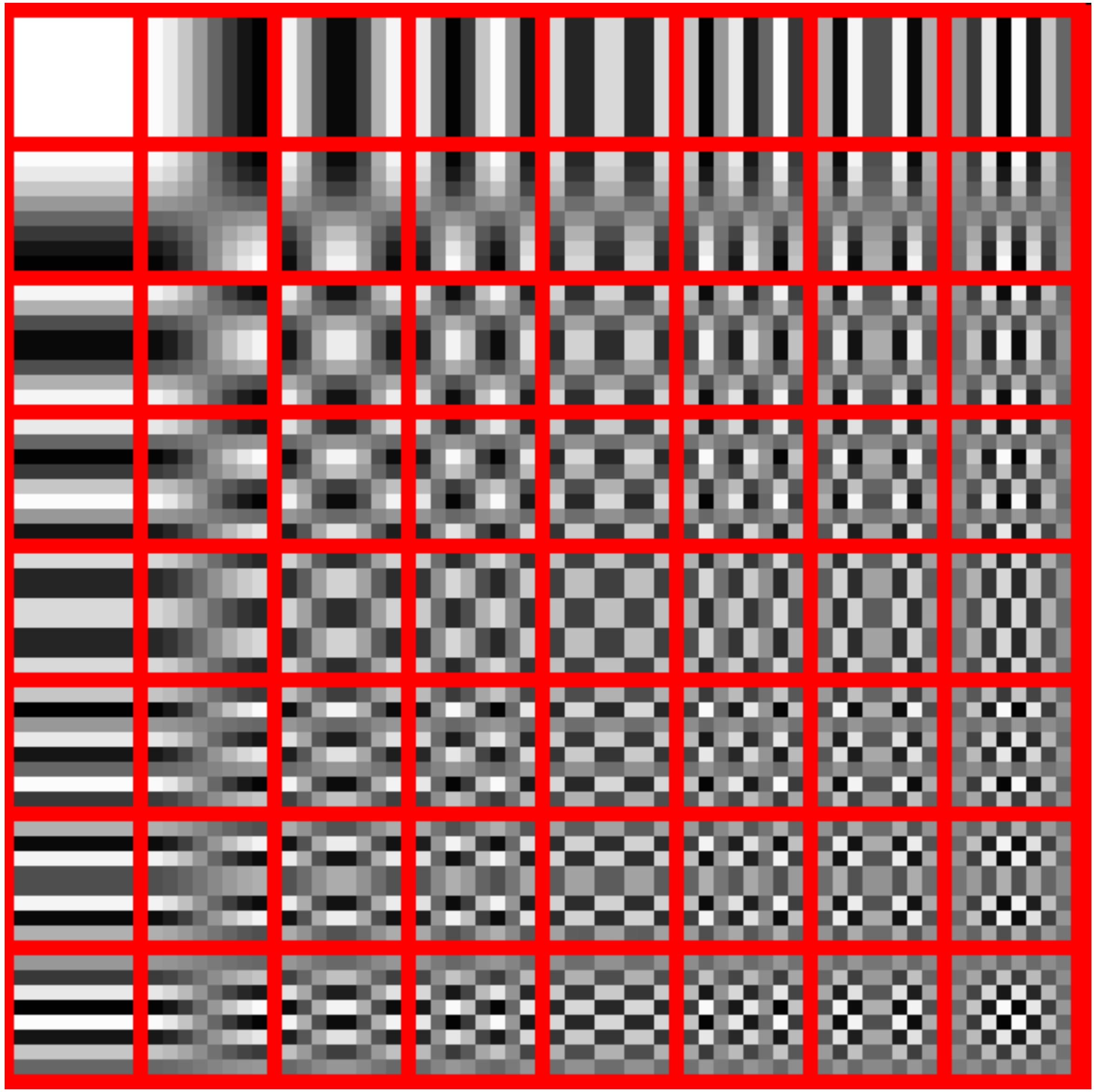


Looks like the discrete cosine bases of JPG!...

dictionary learning



2D Discrete Cosine Basis

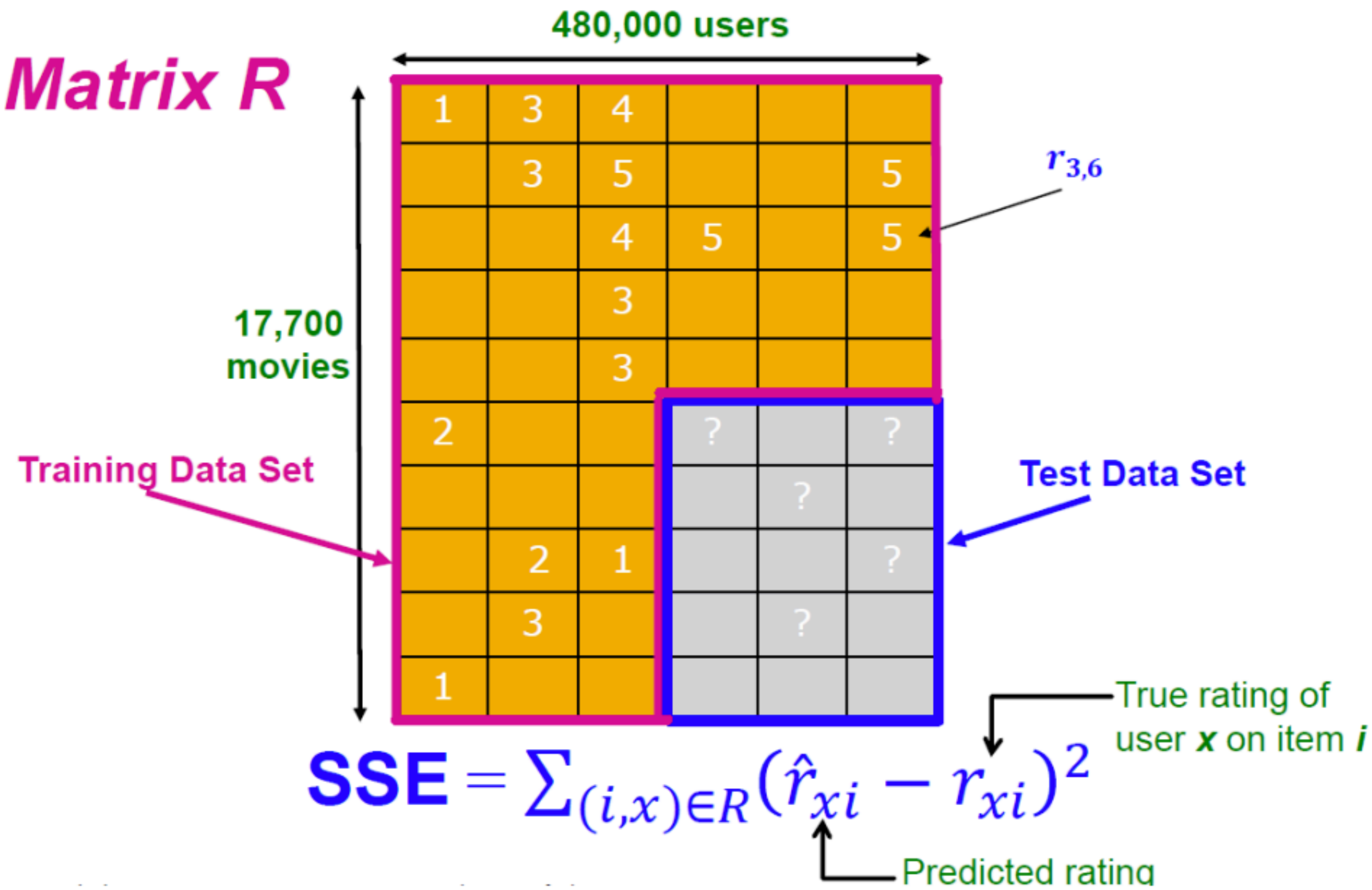


http://en.wikipedia.org/wiki/Discrete_cosine_transform

Dimensionality reduction

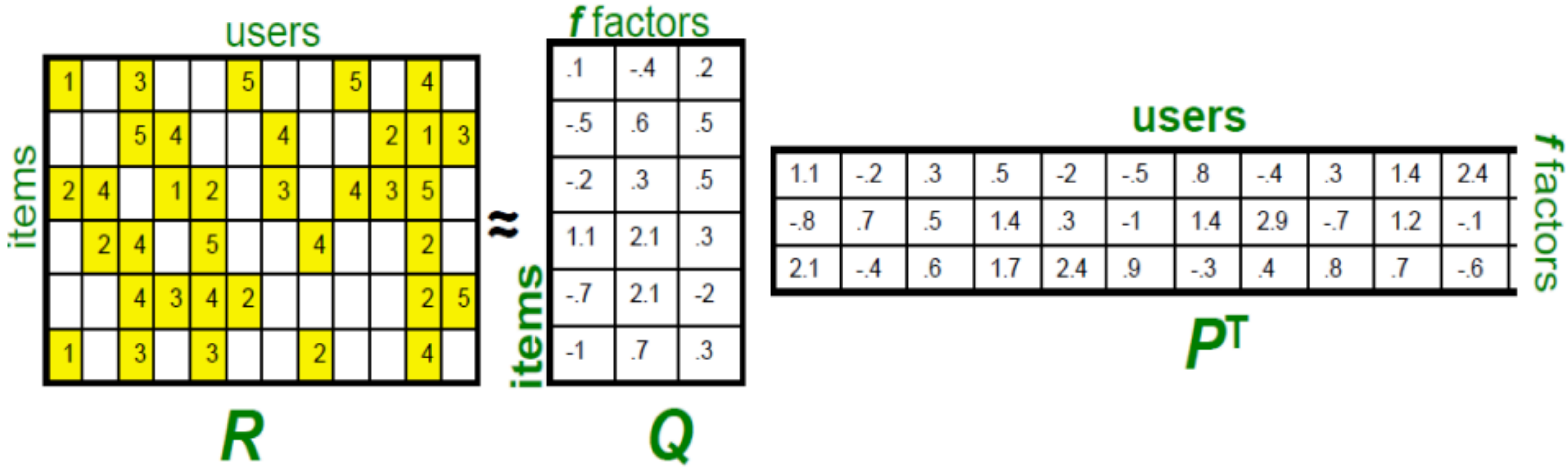
- PCA (Principal Component Analysis):
  - Find projection that maximize the variance
- ICA (Independent Component Analysis):
  - Very similar to PCA except that it assumes non-Guassian features
- Multidimensional Scaling:
  - Find projection that best preserves inter-point distances
- LDA(Linear Discriminant Analysis):
  - Maximizing the component axes for class-separation
- …
- …

Netflix Competition



Latent Factors as Low rank matrix

- low rank factorization on Netflix data: $R \approx Q \cdot P^T$



Math behind Netflix

- Matrix $M \in \mathbb{R}^{n_1 \times n_2}$
- Observe subset of entries
- Can we guess the missing entries?

$$\begin{bmatrix} \times & ? & ? & ? & \times & ? \\ ? & ? & \times & \times & ? & ? \\ \times & ? & ? & \times & ? & ? \\ ? & ? & \times & ? & ? & \times \\ \times & ? & ? & ? & ? & ? \\ ? & ? & \times & \times & ? & ? \end{bmatrix}$$

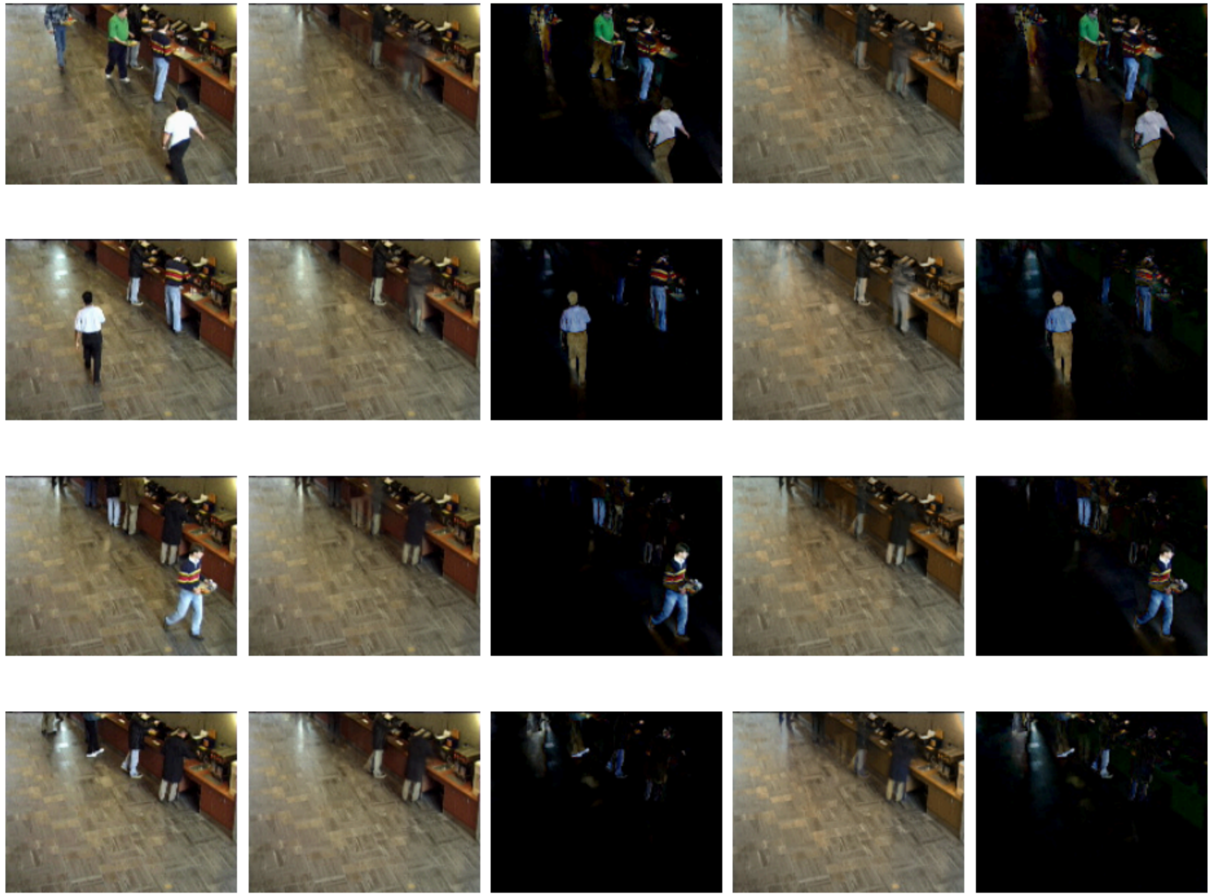
Hope: only one low-rank matrix consistent with the sampled entries

Recovery by minimum complexity

$$\begin{array}{ll} \text{minimize} & \text{rank}(X) \\ \text{subject to} & X_{ij} = M_{ij}, \quad (i,j) \in \Omega \end{array}$$

Another application

- Partition the video into moving and static parts



- Math behind:
  - Change smallest number of pixels (people), make the matrix low rank (background)