Linear Algebra - Problem Set 5 - Solutions

Exercise I ($6 \times 5 = 30$ points)

1. If $B = M^{-1}AM$, why is $\det B = \det A$?
   $\det B = \det(M^{-1}AM) = \det M^{-1} \det A \det M = \frac{1}{\det M} \det A \det M = \det A$
2. If the entries of A and A^{-1} are all integers, What are all the possible values of $\det A$ and $\det A^{-1}$?
   If the entries of A and A^{-1} are all integers, then $\det A$ and $\det A^{-1}$ must be integers.
   But $\det A^{-1} = \dfrac{1}{\det A} \implies \det A$ and $\det A^{-1}$ can only be ± 1.
3. Suppose λ is an eigenvalue of A. Prove the following:
   (a) If A is invertible, then λ^{-1} is an eigenvalue of A^{-1}.
   Suppose A is invertible and λ is an eigenvalue of A, with corresponding eigen vector $\vec{v}$. Then
   $$A^{-1}(\lambda\vec{v}) = \vec{v} = \frac{1}{\lambda}(\lambda\vec{v}).$$
   (b) $\lambda + c$ is an eigenvalue of $A + cI$, where c is any constant.
   Suppose λ is an eigenvalue of A with corresponding eigenvector $\vec{v}$ and c is any constant. Then
   $$(A + cI)\vec{v} = A\vec{v} + cI\vec{v} = \lambda\vec{v} + c\vec{v} = (\lambda + c)\vec{v}.$$
   So $\lambda + c$ is an eigenvector of $A + cI$.
   (c) A and A^T have the same eigenvalues.
   Suppose λ is an eigenvalue of A. Note that $(A - \lambda I)^T = A^T - \lambda I$. Then,
   $$\det\left(A^T - \lambda I\right) = \det\left((A - \lambda I)^T\right) = \det(A - \lambda I).$$
   Thus A and A^T have the same characteristic polynomials, hence they have the same eigenvalues.
4. Suppose a matrix A satisfies $A^2 = I$. Prove that the eigenvalues of A must be 1 or -1.
   Consider any eigenvalue λ of A, and let $\vec{x}$ be an arbitrary eigenvector of A corresponding to λ. Hence, we have:
   $$A\vec{x} = \lambda\vec{x} \Rightarrow A^2\vec{x} = \lambda A\vec{x} \Rightarrow I\vec{x} = \lambda A\vec{x} \Rightarrow \vec{x} = \lambda A\vec{x}$$
   Note that $\lambda(A\vec{x}) = \lambda(\lambda\vec{x}) = \lambda^2\vec{x}$. Hence, we have$\vec{x} = \lambda^2\vec{x}$. As $\vec{x}$ is not $\vec{0}$ (eigenvectors are nonzero), it follows that $\lambda^2 = 1$ and thus $\lambda = \pm 1$

Exercise II ($2 \times 10 = 20$ points)

1. Compute the eigenvector of $\begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & 0 & 1 \end{bmatrix}$ corresponding to eigenvalue -3.

   Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & 0 & 1 \end{bmatrix}$, and compute $(A + 3I)\vec{x} = \vec{0}$, i.e.,
   $$\begin{bmatrix} 4 & 0 & -1 \\ 1 & 0 & 0 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} \boxed{1} & 0 & 0 \\ 4 & 0 & -1 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
   $$\implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \boxed{-1} \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
   The above system gives that $x_1 = x_3 = 0$ and x_2 is a free variable.

   Hence, an eigenvector corresponding to the eigenvalue -3 is given by $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

2. Choose the third row of $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ & & \end{bmatrix}$, such that the characteristic polynomial is $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$.

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & 1-\lambda & 1 \\ a & b & c-\lambda \end{vmatrix} = -\lambda\big[(1-\lambda)(c-\lambda) - b\big] - \big[c - \lambda - a\big] = -\lambda^3 + (1+c)\lambda^2 + (1+b-c)\lambda + (a-c)$$

$1 + c = 4 \implies \boxed{c = 3}$ $\quad 1 + b - c = 5 \implies \boxed{b = 7}$ $\quad a - c = 6 \implies \boxed{a = 9}$

Exercise III ($5 \times 5 = 25$ points)

1. Compute the eigenvalues of A.

   $\det(A - \lambda I) = 0 \implies \begin{vmatrix} 0.4-\lambda & 0.3 \\ 0.2 & 0.5-\lambda \end{vmatrix} = 0 \implies (0.4-\lambda)(0.5-\lambda) - 0.06 = 0 \implies \lambda_1 = 0.2,\ \lambda_2 = 0.7$

2. Compute the eigenvectors associated with each of the eigenvalues in A.

   - $\lambda_1 = 0.2$: Solve $[A - 0.2I]\vec{x}_1 = \vec{0} \implies \left[\begin{array}{cc|c} 0.2 & 0.3 & 0 \\ 0.2 & 0.3 & 0 \end{array}\right]$

     Let $\vec{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and note that $x_1 = -\frac{3}{2}x_2$. Therefore, $\vec{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}x_2 \\ x_2 \end{bmatrix} = \frac{1}{2}x_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ Choose $\vec{x}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, and note that any multiple of this vector is also an eigenvector.
   - $\lambda_1 = 0.7$: Solve $[A - 0.7I]\vec{x}_1 = \vec{0} \implies \left[\begin{array}{cc|c} -0.3 & 0.3 & 0 \\ 0.2 & -0.2 & 0 \end{array}\right]$. Let $\vec{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and note that $x_1 = x_2$. Therefore, $\vec{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Choose $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and note that any multiple of this vector is also an eigenvector.

3. Use your results in (a) and (b) to diagonalize A.

$$A = X\Lambda X^{-1} = \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0.2 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0.2 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} -1/5 & 1/5 \\ 2/5 & 3/5 \end{bmatrix}$$

4. Find a formula for A^k.

$$A^k = X\Lambda^k X^{-1} = \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0.2^k & 0 \\ 0 & 0.7^k \end{bmatrix} \begin{bmatrix} -1/5 & 1/5 \\ 2/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 3/5(0.2^k) + 2/5(0.7^k) & -3/5(0.2^k) + 3/5(0.7^k) \\ -2/5(0.2^k) + 2/5(0.7^k) & 2/5(0.2^k) + 3/5(0.7^k) \end{bmatrix}$$

5. Use your result in (d) to find compute the matrix A^7. What is the limit of A as $k \to \infty$?

$$A^7 = \begin{bmatrix} 3/5(0.2^7) + 2/5(0.7^7) & -3/5(0.2^7) + 3/5(0.7^7) \\ -2/5(0.2^7) + 2/5(0.7^7) & 2/5(0.2^7) + 3/5(0.7^7) \end{bmatrix} \qquad \lim_{k\to\infty} A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Exercise IV (10 points)

Show that the matrices A and B below are similar by finding M so that $B = M^{-1}AM$:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}.$$

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want an *invertible* matrix M that satisfies the relation $A = MBM^{-1} \implies AM = MB$. We get:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \implies \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} 4a+2b & 3a+b \\ 4c+2d & 3c+d \end{bmatrix}.$$

$$\begin{aligned} a + 2c &= 4a + 2b \\ b + 2d &= 3a + b \\ 3a + 4c &= 4c + 2d \\ 3b + 4d &= 3c + d \end{aligned}$$

$\implies \boxed{3a = 2d}$ and $\boxed{b = c - d}$. To make M invertible, let $c = 1$ and $d = 3$, then $M = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$. Notice that $\det M = 8$, which means that M is invertible, and therefore A and B are similar. **Note**: Other choices of c and d could also work, as long as the resulting matrix M is invertible. Always be sure to check that $A = MBM^{-1}$ is valid for your choice of M.

Exercise V (15 points)

Diagonalize the matrix $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ into $Q\Lambda Q^T$, where Q is an orthogonal matrix.

First, show work and verify that the eigenvalues of A are $7, -2$.

Then, we compute the eigenvectors associated with the given eigenvalues.

- $\underline{\lambda = 7}$: Solve $[A - 7I]\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} -4 & -2 & 4 & 0 \\ -2 & -2 & 2 & 0 \\ 4 & 2 & -4 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} -4 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]. \text{ This gives } -4x_1 - 2x_2 + 4x_3 = 0 \Longrightarrow \boxed{x_1 = -\frac{1}{2}x_2 + x_3}.$$

Then, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2}x_2 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and we choose the eigenvectors $\vec{x}_1$ and $\vec{x}_2$ associated with $\lambda = 7$ to be

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \qquad \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

- $\underline{\lambda = -2}$: Solve $[A + 2I]\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 5 & -2 & 4 & 0 \\ -2 & 8 & 2 & 0 \\ 4 & 2 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -2 & 8 & 2 & 0 \\ 5 & -2 & 4 & 0 \\ 4 & 2 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 4 & 1 & 0 \\ 5 & -2 & 4 & 0 \\ 4 & 2 & 5 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 4 & 1 & 0 \\ 0 & 18 & 9 & 0 \\ 0 & 18 & 9 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 4 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This gives $\boxed{x_1 = -x_3}$ and $\boxed{x_2 = -\frac{1}{2}x_3}$. Then, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{2}x_3 \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$, and we choose the eigenvector $\vec{x}_3$ associated with $\lambda = -2$ to be $\vec{x}_3 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$. Before diagonalizing A, we first make the eigenvectors orthogonal using Gram-Schmidt. $\vec{x}_1$ and $\vec{x}_3$ are already orthogonal, and so are $\vec{x}_2$ and $\vec{x}_3$. However, $\vec{x}_1$ and $\vec{x}_2$ are not , so:

Let $\vec{q}_1 = \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, then*

$$\vec{q}_2 = \vec{x}_1 - \frac{\vec{x}_1^T \vec{q}_1}{\vec{q}_1^T \vec{q}_1}\vec{q}_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 2 \\ 1/2 \end{bmatrix}$$

Normalizing $\vec{q}_1$ and $\vec{q}_2$, we get: $\vec{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$, $\quad \vec{q}_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$ and normalizing $\vec{x}_3$, we get: $\vec{q}_3 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$

Thus, the diagonalization (eigen-decomposition) of A gives:

$$A = Q\Lambda Q^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix} \begin{bmatrix} 7 & & \\ & 7 & \\ & & -2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{18} & 4/\sqrt{18} & 1/\sqrt{18} \\ -2/3 & -1/3 & 2/3 \end{bmatrix}$$

*Note to the student and grader: can start with $\vec{q}_1 = \vec{x}_1$, in that case the matrix Q below will have swapped columns 1 and columns 2, and the matrix Q^{-1} will be different. Check by verifying that indeed $A = Q\Lambda Q^T$.