Linear Algebra - Problem Set 4 - Solutions

Exercise I ($3 \times 5 = 15$ points)

Consider the vectors $\vec{a}_1 = (-1, 2, 2)$, $\vec{a}_2 = (2, 2, -1)$, and $\vec{a}_3 = (2, -1, 2)$.

1. Compute the projection matrices P_1 and P_2 onto the lines in the direction of $\vec{a}_1$ and $\vec{a}_2$, respectively. Multiply those projection matrices and explain why their product is what it is.

$$P_1 = \frac{\vec{a}_1 \vec{a}_1^T}{\vec{a}_1^T \vec{a}_1} = \begin{bmatrix} 1/9 & -2/9 & -2/9 \\ -2/9 & 4/9 & 4/9 \\ -2/9 & 4/9 & 4/9 \end{bmatrix}$$

$$P_2 = \frac{\vec{a}_2 \vec{a}_2^T}{\vec{a}_2^T \vec{a}_2} = \begin{bmatrix} 4/9 & 4/9 & -2/9 \\ 4/9 & 4/9 & -2/9 \\ -2/9 & -2/9 & 1/9 \end{bmatrix}$$

$$\implies P_1 P_2 = \begin{bmatrix} 1/9 & -2/9 & -2/9 \\ -2/9 & 4/9 & 4/9 \\ -2/9 & 4/9 & 4/9 \end{bmatrix} \begin{bmatrix} 4/9 & 4/9 & -2/9 \\ 4/9 & 4/9 & -2/9 \\ -2/9 & -2/9 & 1/9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $\vec{a}_1$ and $\vec{a}_2$ are orthogonal, then projecting onto $\vec{a}_1$ followed by projecting onto $\vec{a}_2$, or vice versa, leads to $\vec{0}$.

2. Find the projection vectors $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$ of $\vec{b} = (1, 0, 0)$ onto the lines in the direction of $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$. Add the three projections $\vec{p}_1 + \vec{p}_2 + \vec{p}_3$. What do you notice? Why does this make sense?

$$\vec{p}_1 = P_1 \vec{b} = \begin{bmatrix} 1/9 \\ -2/9 \\ -2/9 \end{bmatrix}$$

$$\vec{p}_2 = P_2 \vec{b} = \begin{bmatrix} 4/9 \\ -2/9 \\ -2/9 \end{bmatrix}$$

$$\vec{p}_3 = \frac{\vec{b}^T \vec{a}_3}{\vec{a}_3^T \vec{a}_3} \vec{a}_3 = \begin{bmatrix} 4/9 \\ -2/9 \\ 4/9 \end{bmatrix}$$

Note that $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = \vec{b}$ which makes sense since the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are orthogonal so they span $\mathbb{R}^3$.

3. Find the projection matrix P_3 onto the line directed by $\vec{a}_3$, then find $P_1 + P_2 + P_3$. Comment on the result (explain why it makes sense).

$$P_3 = \frac{\vec{a}_3 \vec{a}_3^T}{\vec{a}_3^T \vec{a}_3} = \begin{bmatrix} 4/9 & -2/9 & 4/9 \\ -2/9 & 1/9 & -2/9 \\ 4/9 & -2/9 & 4/9 \end{bmatrix}$$

Note that $P_1 + P_2 + P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$.

This is because $\vec{p}_i = \vec{b} P_i$ so

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = \vec{b} \implies \vec{b} P_1 + \vec{b} P_2 + \vec{b} P_3 = \vec{b} \implies \vec{b}(P_1 + P_2 + P_3) = \vec{b}$$

so $P_1 + P_2 + P_3 = I$.

Exercise II ($2 \times 10 = 20$ points)

1. Find the projection of $\vec{b} = (4, 3, 1, 0)$ onto the nullspace of the matrix
$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 2 \end{bmatrix}.$$

First, we proceed by finding the nullspace of A, i.e., we solve the system $A\vec{x} = \vec{0}$, where $\vec{x} = (x_1, x_2, x_3, x_4)$.
$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} \boxed{1} & 0 & -1 & -3 & 0 \\ 0 & \boxed{1} & 2 & 4 & 0 \end{array}\right]$$
The last step involved writing A in RREF, which simplifies the computation of the nullspace vectors, as it always leads to the pivot variables being written in terms of the free variables. Note that since the third and fourth column do not contain any pivots, then they are free columns, which means that the variables x_3 and x_4 are free.

Row 1: $x_1 = x_3 + 3x_4$ and Row 2: $x_2 = -2x_3 - 4x_4$

Therefore, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \end{bmatrix} \implies \beta_{\text{Nul}A} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}.$

Now, let $B = \begin{bmatrix} 1 & 3 \\ -2 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, and project the vector $\vec{b} = \begin{bmatrix} 4 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ onto the column space of B, i.e., solve
$$B^T B\hat{\vec{x}} = B^T\vec{b} \implies \begin{bmatrix} 6 & 11 \\ 11 & 26 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \implies \begin{bmatrix} 6 & 11 \\ 0 & 35/6 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 11/6 \end{bmatrix}$$
Therefore, we find that $\boxed{\hat{x}_1 = -26/35}$ and $\boxed{\hat{x}_2 = 11/35}$. The projection of $\vec{b}$ onto the column space of B is then
$$\vec{p} = B\hat{\vec{x}} = \begin{bmatrix} 1 & 3 \\ -2 & -4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -26/35 \\ 11/35 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 8/35 \\ -26/35 \\ 11/35 \end{bmatrix}$$

2. Find the best approximation to $\vec{u} = (3, -7, 2, 3)$ as a linear combination of $\vec{v}_1 = (2, -1, -3, 1)$ and $\vec{v}_2 = (1, 1, 0, -1)$.
$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 \implies \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \Leftrightarrow$$
$$\underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 1 \\ -3 & 0 \\ 1 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix}}_{\vec{b}}$$
The above system has no solution, we find an approximate solution $\hat{\vec{x}}$ using least squares.
$$A^T A\hat{\vec{x}} = A^T\vec{b},$$
$$\begin{bmatrix} 2 & -1 & -3 & 1 \\ 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ -3 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -3 & 1 \\ 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix}$$
$$\begin{bmatrix} 15 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \end{bmatrix}$$
This gives $\boxed{\hat{x}_1 = 2/3}$, which is the best approximation for c_1, and $\boxed{\hat{x}_2 = -7/3}$, which is the best approximation for c_2. Hence, the best approximation of $\vec{u}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$ is
$$\hat{x}_1\vec{v}_1 + \hat{x}_2\vec{v}_2 = \begin{bmatrix} -1 \\ -3 \\ -2 \\ 3 \end{bmatrix}.$$

Exercise III ($2 \times 10 = 20$ points)

1. Consider the data points $(-2,3)$, $(2,1)$, $(3,-4)$, and $(5,2)$. Find the equation of the *best fit line* $b = C + Dt$.
   Using the given data points in the linear equation $b = C + Dt$, we get
$$\begin{aligned} 3 &= C - 2D \\ 1 &= C + 2D \\ -4 &= C + 3D \\ 2 &= C + 5D \end{aligned}$$
   which can be written in matrix-vector form as
$$\begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}.$$
   This system does not have a solution. Therefore, we find an approximate solution using least squares. Hence, we solve
$$A^T A\vec{x} = A^T\vec{b}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$$
$$\implies \begin{bmatrix} 4 & 8 \\ 8 & 42 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$$
$$\implies \begin{bmatrix} 4 & 8 \\ 0 & 26 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$$
   Therefore, $\hat{x}_1 = 33/26$ and $\hat{x}_2 = -10/26$. Note that $\hat{x}_1$ is an approximation for the intercept C and $\hat{x}_2$ is an approximation for the slope D. Thus, the least-squares fit gives
$$b = \frac{33}{26} - \frac{10}{26}t.$$

2. What if instead, for part 2., you had found the best-fit line $t = C' + D'b$. Is the slope $\bar{D}$ related to the slope found in 1? How would you have expected the slopes to be related?
   Similarly, we get
$$\begin{aligned} -2 &= \bar{C} + 3\bar{D} \\ 2 &= \bar{C} + \bar{D} \\ 3 &= \bar{C} - 4\bar{D} \\ 5 &= \bar{C} + 2\bar{D} \end{aligned}$$
$$\begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 1 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \bar{C} \\ \bar{D} \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 3 \\ 5 \end{bmatrix}.$$
$$A\vec{x} = \vec{b} \implies A^T A\vec{x} = A^T\vec{b}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 1 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & -4 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 3 \\ 5 \end{bmatrix}$$
$$\implies \begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 8 \\ -6 \end{bmatrix}$$
$$\implies \begin{bmatrix} 2 & 30 \\ 0 & -56 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} -6 \\ 20 \end{bmatrix}$$
   Therefore, $\hat{x}_1 = 63/29$ and $\hat{x}_2 = -10/29$. Note that $\hat{x}_1$ is an approximation for the intercept $\bar{C}$ and $\hat{x}_2$ is an approximation for the slope $\bar{D}$. Thus, the least-squares fit gives
$$t = \frac{63}{29} - \frac{10}{29}b.$$
   Note that the slopes D and $\bar{D}$ are not reciprocals of each other, as we would expect the case to be in theory. This is due to the fact that the least-squares solutions in (a) and (b) are minimizing different errors, and hence the approximations lead to different results.

Exercise IV $(2 \times 10 = 20$ points)

The given set is a basis for a subspace W. Use the Gram-Schmidt process to produce an orthonormal basis for W.

1. $\left\{ \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} \right\}$

   To find an orthonormal set of basis vectors of W, we use the Gram-Schmidt process.
   We need to find two orthonormal vectors, $\vec{q}_1$ and $\vec{q}_2$ that span the same space as $\vec{a}_1$ and $\vec{a}_2$. We follow two G-S steps:

   - Let $\vec{\sigma}_1 = \vec{a}_1$. Then $\boxed{\vec{q}_1 = \frac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix}}$
   - For the second vector, subtract from $\vec{a}_2$ its projection onto span$(\vec{\sigma}_1)$:

   $$\vec{\sigma}_2 = \vec{a}_2 - \frac{\vec{\sigma}_1^T \vec{a}_2}{\vec{\sigma}_1^T \vec{\sigma}_1} \vec{\sigma}_1 = \begin{bmatrix} 7 \\ -7 \\ -4 \\ 1 \end{bmatrix} - \frac{36}{18} \begin{bmatrix} 1 \\ -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix}. \text{Then, } \boxed{\vec{q}_2 = \frac{1}{\sqrt{43}} \begin{bmatrix} 5 \\ 1 \\ -4 \\ -1 \end{bmatrix}}$$

2. $\left\{ \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right\}$

   We need to find three orthonormal vectors, $\vec{q}_1, \vec{q}_2$ and $\vec{q}_3$ that span the same space as $\vec{a}_1, \vec{a}_2$ and $\vec{a}_3$. We follow three G-S steps:

   - Let $\vec{\sigma}_1 = \vec{a}_1$. Then $\boxed{\vec{q}_1 = \frac{1}{\sqrt{20}} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}}$
   - For the second vector, subtract from $\vec{a}_2$ its projection onto span$(\vec{\sigma}_1)$:

   $$\vec{\sigma}_2 = \vec{a}_2 - \frac{\vec{\sigma}_1^T \vec{a}_2}{\vec{\sigma}_1^T \vec{\sigma}_1} \vec{\sigma}_1 = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \frac{10}{20} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}. \text{Then, } \boxed{\vec{q}_2 = \frac{1}{\sqrt{105}} \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}}$$

   - For the third vector, subtract from $\vec{a}_3$ its projection onto span$(\vec{\sigma}_1, \vec{\sigma}_2)$:

   $$\vec{\sigma}_3 = \vec{a}_3 - \frac{\vec{\sigma}_1^T \vec{a}_3}{\vec{\sigma}_1^T \vec{\sigma}_1} \vec{\sigma}_1 - \frac{\vec{\sigma}_2^T \vec{a}_3}{\vec{\sigma}_2^T \vec{\sigma}_2} \vec{\sigma}_2 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - \frac{10}{20} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} - \frac{25}{105} \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix}. \tag{1}$$

   Then, $\boxed{\vec{q}_3 = \frac{1}{\sqrt{21}} \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix}}$

Exercise V (20 + 5 = 25 points)

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. Compute the QR factorization of A.

   Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ be the (consecutive) columns of the matrix A, and let

$$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}$$

   be the first column of the matrix Q. Then, $\vec{q}_2 = \vec{v}_2 - \vec{p}_{21}$, where $\vec{p}_{21}$ is the orthogonal projection of $\vec{v}_2$ onto $\vec{q}_1$, i.e.,

$$\begin{aligned} \vec{q}_2 &= \vec{v}_2 - \frac{\vec{v}_2^T \vec{q}_1}{\vec{q}_1^T \vec{q}_1} \vec{q}_1 \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ 0 \end{bmatrix} \end{aligned}$$

   To find $\vec{q}_3$, we must remove $\vec{p}_{32}$ and $\vec{p}_{31}$ from $\vec{v}_3$, where $\vec{p}_{32}$ is the projection of $\vec{v}_3$ onto $\vec{q}_2$, and $\vec{p}_{31}$ is the projection of $\vec{v}_3$ onto $\vec{q}_1$ i.e.,

$$\vec{q}_3 = \vec{v}_3 - \vec{p}_{32} - \vec{p}_{31}$$

$$\begin{aligned} \vec{q}_3 &= \vec{v}_3 - \frac{\vec{v}_3^T \vec{q}_2}{\vec{q}_2^T \vec{q}_2} \vec{q}_2 - \frac{\vec{v}_3^T \vec{q}_1}{\vec{q}_1^T \vec{q}_1} \vec{q}_1 \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ 0 \end{bmatrix}}{\begin{bmatrix} \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & 0 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ 0 \end{bmatrix}} \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{\sqrt{6}} \begin{bmatrix} \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ 0 \end{bmatrix} - 0 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -1/3 \\ -2/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 1/3 \\ 1 \end{bmatrix} \xrightarrow{\text{normalize}} \begin{bmatrix} \sqrt{3}/6 \\ -\sqrt{3}/6 \\ \sqrt{3}/6 \\ \sqrt{3}/2 \end{bmatrix} \end{aligned}$$

Thus, the matrix Q is given by

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{\sqrt{3}}{6} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-\sqrt{3}}{6} \\ 0 & \frac{2}{\sqrt{6}} & \frac{\sqrt{3}}{6} \\ 0 & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}$$

and we can compute R as

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{\sqrt{3}}{6} & \frac{-\sqrt{3}}{6} & \frac{\sqrt{3}}{6} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{3}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ 0 & 0 & \frac{2\sqrt{3}}{3} \end{bmatrix}$$

Hence, $A = QR \implies A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{\sqrt{3}}{6} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-\sqrt{3}}{6} \\ 0 & \frac{2}{\sqrt{6}} & \frac{\sqrt{3}}{6} \\ 0 & 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{3}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ 0 & 0 & \frac{2\sqrt{3}}{3} \end{bmatrix}.$

2. Use your result to find a factorization of $A^T A$. Can this be viewed as an LU factorization? Why or why not?

$$A^T A = (QR)^T (QR) = R^T Q^T Q R = R^T R$$

Since R is upper triangular, then R^T is lower triangular, and hence this would correspond to something resembling the LU decomposition of the matrix A. This is not exactly the LU decomposition though, because the lower triangular matrix does not have diagonal entries equal to 1.

Note: This factorization (of symmetric matrices) is in fact known as the Cholesky decomposition, and it used extensively in applications, especially for efficient numerical solutions.