Linear Algebra - Problem Set 3 - Solutions

Exercise I ($3 \times 5 = 15$ points)

The following statements **are false**. Clearly explain why.

1. Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. Define the usual addition of elements of V coordinatewise, but define scalar multiplication differently: for $c \in \mathbb{R}$ and $(a_1, a_2) \in \mathbb{R}^2$, define $c(a_1, a_2) = (0, a_2)$. V is a vector space over $\mathbb{R}^2$ with these operations.
   Let $\vec{v} = (v_1, v_2) \in V$, then the property (VS5) is violated since for each $\vec{v} \in V$, $1\vec{v} = \vec{v}$. Here, $1\vec{v} = (0, v_2) \neq \vec{v}$, and hence (VS5) is violated.

2. The system below is solvable as long as $b_2 = 2b_1$ only, and the column space is a plane in $\mathbb{R}^3$.
$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$
   We start by writing the system in augmented form, and then we proceed by performing elimination, i.e.,
$$\left[\begin{array}{ccc|c} \boxed{1} & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 \end{array}\right].$$
   Therefore, for the system to be solvable, we must have:
$$b_2 - 2b_1 = 0 \implies b_2 = 2b_1$$
$$b_3 + b_1 = 0 \implies b_3 = -b_1$$
   The column space is a line in $\mathbb{R}^3$.

3. Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$. Then the variables x_2 and x_4 in a vector $\vec{x} = (x_1, x_2, x_3, x_4) \in \text{Nul}A$ are free variables
$$\begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 2 & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
   The first and second columns are pivot columns, whereas the third and fourth columns are free. Thus, the variables x_3 and x_4 in a vector $\vec{x} = (x_1, x_2, x_3, x_4) \in \text{Nul}A$ are free variables.

Exercise II ($3 \times 5 = 15$ points)

Are the following sets subspaces of $\mathbb{R}^3$ under usual addition and scalar multiplication defined on $\mathbb{R}^3$.? Explain clearly.
Note: 2 to 5 points taken off (total) if the vectors are not clearly defined and the setup/presentation of the proof is not legible.

1. $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 2a_3 \text{ and } a_2 = -7a_3\}$.
   Let $\vec{u} = (u_1, u_2, u_3) \in W_1 \implies \vec{u} = (2u_3, -7u_3, u_3) = u_3(2, -7, 1)$ and $\vec{v} = (v_1, v_2, v_3) \in W_1 \implies \vec{v} = v_3(2, -7, 1)$

   (1) $\vec{u} + \vec{v} = (2u_3 + 2v_3, -7u_3 - 7v_3, u_3 + v_3) \implies \vec{u} + \vec{v} = (2(u_3 + v_3), -7(u_3 + v_3), u_3 + v_3) = (u_3 + v_3)(2, -7, 1) \in W1$
   (2) $c\vec{v} = cv_3(2, -7, 1) \in W_1$
   (3) $\vec{0} = (0, 0, 0) = (2 \cdot 0, -7 \cdot 0, 1 \cdot 0) \in W_1$

   Therefore, W_1 is a subspace of $\mathbb{R}^3$. Note that to receive full credit, you have to verify **all three** properties.

2. $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 4a_2 + 5a_3 = 3\}$.
   Let $\vec{u} = (u_1, u_2, u_3) \in W_2 \implies \vec{u} = \left(\frac{3}{2} + 2u_2 - \frac{5}{2}u_3,\ u_2,\ u_3\right)$ and $\vec{v} = (v_1, v_2, v_3) \in W_2 \implies \vec{v} = \left(\frac{3}{2} + 2v_2 - \frac{5}{2}v_3,\ v_2,\ v_3\right)$

   (1) $\vec{u} + \vec{v} = \left(3 + 2(u_2 + v_2) - \frac{5}{2}(u_3 + v_3),\ u_2 + v_2,\ u_3 + v_3\right) \notin W_2$

(2) $c\vec{v} = c\left(\frac{3}{2} + 2v_2 - \frac{5}{2}v_3,\, v_2,\, v_3\right) = \left(\frac{3c}{2} + 2cv_2 - \frac{5c}{2}v_3,\, cv_2,\, cv_3\right) \notin W_2$

(3) $\vec{0} = (0,0,0) \neq \left(\frac{3}{2}, 0, 0\right) \implies (0,0,0) \notin W_2$

Therefore, W_2 is not a subspace of $\mathbb{R}^3$.

Here, you will receive full credit if you show that either (1), (2), or (3) are not satisfied. You don't have to show that all three properties do not work.

3. $W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 4a_2 + 5a_3 = 0\}$.

Let $\vec{u} = (u_1, u_2, u_3) \in W_3 \implies \vec{u} = \left(2u_2 - \frac{5}{2}u_3,\, u_2,\, u_3\right)$ and $\vec{v} = (v_1, v_2, v_3) \in W_3 \implies \vec{v} = \left(2v_2 - \frac{5}{2}v_3,\, v_2,\, v_3\right)$

(1) $\vec{u} + \vec{v} = \left(2(u_2 + v_2) - \frac{5}{2}(u_3 + v_3),\, (u_2 + v_2),\, (u_3 + v_3)\right) \in W_3$

(2) $c\vec{v} = c\left(2v_2 - \frac{5}{2}v_3,\, v_2,\, v_3\right) = \left(2cv_2 - \frac{5}{2}cv_3,\, cv_2,\, cv_3\right) \in W_3$

(3) $\vec{0} = (0,0,0) = \left(2 \cdot 0 - \frac{5}{2} \cdot 0,\, 0,\, 0\right) \in W_3$

Therefore, W_3 is a subspace of $\mathbb{R}^3$. Note that to receive full credit, you have to verify **all three** properties.

Exercise III (10 points)

Find the complete solution $\vec{x} = \vec{x}_p + \vec{x}_n$ of the linear system below. Clearly label the vectors $\vec{x}_p$ and $\vec{x}_n$.

$$\begin{aligned} x_1 + 3x_2 + x_3 + 2x_4 &= 1 \\ 2x_1 + 6x_2 + 4x_3 + 8x_4 &= 3 \\ 2x_3 + 4x_4 &= 1 \end{aligned}$$

Let $\vec{x} = (x_1, x_2, x_3, x_4)$ and solve $A\vec{x} = \vec{b}$, where $\vec{b} = (1,3,1)$. We write the system in augmented form and then we proceed with elimination, i.e.,

$$\left[\begin{array}{cccc|c} \boxed{1} & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 0 & 0 & \boxed{2} & 4 & 1 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right] \to \left[\begin{array}{cccc|c} \boxed{1} & 3 & 1 & 2 & 1 \\ 0 & 0 & \boxed{2} & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Based on the elimination results, we can see that x_2 and x_4 are free variables.

Row 2: $2x_3 = 1 - 4x_4 \implies x_3 = \frac{1}{2} - 2x_4$

Row 1: $x_1 = 1 - 3x_2 - x_3 - 2x_4 \implies x_1 = \frac{1}{2} - 3x_2$

Thus, we can write the vector $\vec{x}$ as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - 3x_2 \\ x_2 \\ \frac{1}{2} - 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} = \vec{x}_p + \vec{x}_n,$$

where

$$\vec{x}_p = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} \text{ and } \vec{x}_n = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\vec{x} \notin \text{Nul}A, \text{ since } \vec{x} \neq c_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

Exercise IV (5 + 5 + 10 + 10 = 30 points)

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 2 & 4 \\ 4 & 6 & 8 & c \end{bmatrix}.$$

1. Find the matrix R, the row echelon form of the matrix A:

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 1 \\ 2 & 2 & 2 & 4 \\ 4 & 6 & 8 & c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & \boxed{-2} & -4 & 2 \\ 0 & -2 & -4 & c-4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -2 & -4 & 2 \\ 0 & 0 & 0 & c-6 \end{bmatrix}$$

2. What **value** of c gives A a different rank compared to all other values of c? What are the ranks in both cases?
   When $c \neq 6$ there are three pivots and the rank is 3, while when $c = 6$ there are only two pivots and the rank is 2.
3. For each case, find the column space of A.
   The row echelon form is different in these two cases:

   (i) $c = 6$: $R = \begin{bmatrix} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{-2} & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then, there are two pivot columns: the first and second. Hence,

$$\text{Col}A = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix} \right\}$$

   (ii) $c \neq 6$: $R = \begin{bmatrix} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{-2} & -4 & 2 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$. Here, we have divided the last row by $c - 6$ to get a pivot of 1. Then, there are three pivot columns: the first, second, and fourth. Hence,

$$\text{Col}A = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ c \end{bmatrix} \right\}$$

4. For each case, find the nullspace of A.

   (i) $c = 6$: We solve $A\vec{x} = \vec{0}$, which in augmented form (and after elimination) gives

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ 0 & -2 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

   Row 2: $x_2 = -2x_3 + x_4$
   Row 1: $x_1 = -2x_2 - 3x_3 - x_4 = x_3 - 3x_4$

   Thus, we can write the vector $\vec{x}$ as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 3x_4 \\ -2x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

   Therefore,

$$\text{Nul}A = \text{span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

   (ii) $c \neq 6$: We solve $A\vec{x} = \vec{0}$, which in augmented form (and after elimination) gives

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ 0 & -2 & -4 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

Row 3: $x_4 = 0$
Row 2: $x_2 = -2x_3$
Row 1: $x_1 = -2x_2 - 3x_3 = x_3$

Thus, we can write the vector $\vec{x}$ as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore,

$$\text{Nul}A = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Exercise V (10 + 4 × 5 = 30 points)

Let

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 4 & 2 \\ -3 & 6 & 0 & 3 & -9 & -3 \end{bmatrix}$$

1. Determine a basis for the nullspace Nul(A). Work carefully, since you should use this part to answer 2., 3. and 4.

$$\begin{bmatrix} \boxed{1} & -2 & 0 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 4 & 2 \\ -3 & 6 & 0 & 3 & -9 & -3 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & -2 & 0 & -1 & 3 & 1 \\ 0 & 0 & \boxed{1} & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The first and third columns are pivot columns, whereas the second, fourth, fifth, and sixth columns are free. Thus, the variables x_2, x_4, x_5, and x_6 in a vector $\vec{x} = (x_1, x_2, x_3, x_4, x_5, x_6) \in \text{Nul}A$ are free variables.

To find the nullspace, we compute $A\vec{x} = \vec{0}$. In augmented form, this gives

$$\left[\begin{array}{cccccc|c} 1 & -2 & 0 & -1 & 3 & 1 & 0 \\ 2 & -4 & 1 & 0 & 4 & 2 & 0 \\ -3 & 6 & 0 & 3 & -9 & -3 & 0 \end{array}\right] \to \left[\begin{array}{cccccc|c} 1 & -2 & 0 & -1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Row 2: $x_3 = -2x_4 + 2x_5$

Row 1: $x_1 = 2x_2 + x_4 - 3x_5 - x_6$

Thus, we can write the vector $\vec{x}$ as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 - x_6 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence, a basis for Nul$A$ is given by

$$\beta_{\text{Nul}A} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

2. From the information in Part (a), determine the dimensions of the four subspaces $\text{Nul}(A)$, $\text{Col}(A)$, $\text{Col}(A^T)$ and $\text{Nul}(A^T)$.
   $\dim(\text{Nul}(A)) = 4$
   $\dim(\text{Col}(A)) = 2$
   $\dim(\text{Col}(A^T)) = 2$
   $\dim(\text{Nul}(A^T)) = 1$

3. Find a basis for the column space, $\text{Col}(A)$.
   The first and third columns are pivot columns, so a basis for the column space of A is given by

$$\beta_{\text{Col}A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

4. Find a basis for the row space, $\text{Col}(A^T)$.
   We can see in $\text{REF}(A)$ that both the first and second rows have pivots, hence a basis for the row space is

$$\beta_{\text{Col}A^T} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \\ 4 \\ 2 \end{bmatrix} \right\}$$

5. Find a basis for the left nullspace, $\text{Nul}(A^T)$.
   To find the left nullspace, we compute $A^T\vec{x} = \vec{0}$, where $\vec{x} = (x_1, x_2, x_3)$. In augmented form, this gives

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & -3 & 0 \\ -2 & -4 & 6 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ 3 & 4 & -9 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} \boxed{1} & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

   Notice that the first and second columns are pivot columns, whereas the third column is free. Hence, x_3 is a free variable.

   Row 3: $x_2 = 0$
   Row 1: $x_1 = -2x_2 + 3x_3 = 3x_3$

   Thus, we can write the vector $\vec{x}$ as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

   Hence, a basis for $\text{Nul}A^T$ is given by

$$\beta_{\text{Nul}A^T} = \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$