Linear Algebra - Problem Set 2 - Solutions

Exercise I ($5 \times 5 = 25$ points)

True or False? In both cases, explain clearly. A counterexample is good justification in case a statement is False.

1. It is possible for a system $A\vec{x} = \vec{b}$ of equations to have exactly two solutions, e.g., $\vec{x} = \vec{u}$ and $\vec{x} = \vec{v}$.

   FALSE: Let $\vec{x}$ and $\vec{y}$ be solutions to a system of equations, such that $A\vec{x} = \vec{b}$ and $A\vec{y} = \vec{b}$. Taking the sum, we get:

   $$A\vec{x} + A\vec{y} = 2\vec{b} \implies A(\vec{x} + \vec{y}) = 2\vec{b} \implies A\left(\frac{\vec{x} + \vec{y}}{2}\right) = \vec{b}$$

   Therefore, $\frac{\vec{x} + \vec{y}}{2}$ must also be a solution. We showed that there is now a third solution that is distinct from the original ones. Thus the claim is false.

2. A matrix with a column of zeros cannot be invertible.

   TRUE : If one column is a vector of zeros, the linear combination of the columns of A including this column has a nontrivial linear combination that equals zero (try 0 coefficients for all columns except this one, where the coefficient can be any real number). Thus the columns are dependent and A is singular.

3. If every row of a matrix adds up to zero, then the matrix cannot be invertible.

   TRUE: The linear combination of the column vectors where each coefficient is 1 gives a non-trivial solution to the equation $c_1\vec{v_1} + \cdots c_n\vec{v_n} = 0$, thus the columns are linearly dependent and so A is singular.

4. Every matrix with 1's down the main diagonal is invertible.

   FALSE : Consider the matrix below, which has 1's along its main diagonal but is not invertible since the operation $R_2 - R_1 \rightarrow R_2$ yields a row of zeros.

   $$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

5. If A is invertible, then A^{-1} and A^2 are invertible.

   TRUE : If A is invertible, then A^{-1} is the inverse of A and $AA^{-1} = I$. From this, we can also conclude that A^{-1} is invertible with the inverse of A. Next, we consider $(AA)^{-1} = A^{-1}A^{-1}$, which we know exists since A^{-1} exists, to be the inverse of $A^2 = AA$. Then we calculate $AA(AA)^{-1} = A(AA^{-1})A^{-1} = A(I)A^{-1} = I$. Then we have found an inverse and A^2 is invertible.

Exercise II ($3 \times 10 = 30$ points)

Consider the matrices

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \\ -2 & -7 & -9 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

1. Solve the system $A\vec{x} = \vec{b}$, where $\vec{x} = (x_1, x_2, x_3)$ and $\vec{b} = (1, -2, 4)$.

   Solving in augmented form:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 2 & 5 & 6 & -2 \\ -2 & -7 & -9 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & -1 & -1 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & 1 & 10 \end{array}\right]$$

   row 3: $x_3 = 10$

   row 2: $x_2 + 2x_3 = 4 \implies x_2 = -16$

   row 3: $x_1 + 3x_2 + 4x_3 = 1 \implies x_1 = 9$

2. Use elimination to find the inverse of A.

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 3 & 4 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ -2 & -7 & -9 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & \boxed{-1} & -2 & -2 & 1 & 0 \\ 0 & -1 & -1 & 2 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & \boxed{1} & 4 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -15 & 4 & -4 \\ 0 & -1 & 0 & 6 & -1 & 2 \\ 0 & 0 & 1 & 4 & -1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -15 & 4 & -4 \\ 0 & \boxed{1} & 0 & -6 & 1 & -2 \\ 0 & 0 & 1 & 4 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & 2 \\ 0 & 1 & 0 & -6 & 1 & -2 \\ 0 & 0 & 1 & 4 & -1 & 1 \end{array}\right]$$

   Hence $A^{-1} = \begin{bmatrix} 3 & 1 & 2 \\ -6 & 1 & -2 \\ 4 & -1 & 1 \end{bmatrix}$.

3. Compute BC, CB and $A^{-1}CB$. If the product does not exist, explain why.

   It is not possible to compute BC because B is a 2×3 matrix but C is a 2×2 matrix.

   We have $CB = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -10 & 3 \\ 0 & -5 & 4 \end{bmatrix}$

   It is not possible to compute $A^{-1}CB = A^{-1}(CB)$ (by the associative property) because A^{-1} is a 3×3 matrix but BC is a 2×3 matrix.

Exercise III (10 + 5= 15 points)

1. Use elimination (or permutation if needed) matrices to put the following matrix in upper triangular form. At each step, determine which elimination (or permutation) matrix you used.

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ a & b & c & c \\ a & b & c & d \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ a & b & c & d \end{bmatrix}$$

$$E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & b-a & c-a & d-a \end{bmatrix}$$

$$E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad \rightarrow \quad E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & c-b & d-b \end{bmatrix}$$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \rightarrow \quad E_{43}E_{42}E_{32}E_{41}E_{31}E_{21}A = \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

2. Find A's LU factorization. You should be able to conclude this quickly. What are the conditions on a, b, c, d to get $A = LU$ with four pivots?

$$A = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & 0 & c-b & c-b \\ 0 & 0 & 0 & d-c \end{bmatrix}$$

In order for this to work, we require that $\boxed{a \neq 0,\ a \neq b,\ b \neq c,\ c \neq d}$.

Exercise IV (15 points)

Find the LDU decomposition of

$$A = \begin{bmatrix} 1 & -2 & -4 & -3 \\ 2 & -7 & -7 & -6 \\ -1 & 2 & 6 & 4 \\ -4 & -1 & 9 & 8 \end{bmatrix}.$$

Show all the elimination and permutation (if any) matrices used to achieve $A = LDU$.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies E_{21}A = \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & -3 & 1 & 0 \\ -1 & 2 & 6 & 4 \\ -4 & -1 & 9 & 8 \end{bmatrix}, \quad E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies E_{31}E_{21}A = \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ -4 & -1 & 9 & 8 \end{bmatrix}, \quad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{bmatrix} \implies E_{41}E_{31}E_{21}A = \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & -9 & -7 & -4 \end{bmatrix}, \quad E_{41}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} \implies E_{42}E_{41}E_{31}E_{21}A = \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -10 & 1 \end{bmatrix}, \quad E_{42}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}$$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix} \implies E_{43}E_{42}E_{41}E_{31}E_{21}A = \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_{43}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -5 & 1 \end{bmatrix}$$

Therefore, the upper triangular matrix we just obtained can be written as:

$$\begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = DU$$

$$L = (E_{43}E_{42}E_{41}E_{31}E_{21})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{41}^{-1}E_{42}^{-1}E_{43}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -4 & 3 & -5 & 1 \end{bmatrix}$$

Hence, $A = LDU$ gives:

$$\begin{bmatrix} 1 & -2 & -4 & -3 \\ 2 & -7 & -7 & -6 \\ -1 & 2 & 6 & 4 \\ -4 & -1 & 9 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -4 & 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -4 & -3 \\ 0 & 1 & -1/3 & 0 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise V (15 points)

Factor the following system into $A = LU$ or $PA = LU$, and use the factorization to obtain a solution for (x_1, x_2, x_3). Show all steps to receive full credit.

$$\begin{array}{rcrcrcl} 2x_1 & - & x_2 & + & 2x_3 & = & 1 \\ -6x_1 & & & - & 2x_3 & = & 0 \\ 8x_1 & - & x_2 & + & 5x_3 & = & 4 \end{array}$$

Let A be the coefficient matrix, i.e., $A = \begin{bmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{bmatrix}$. We factor this matrix into $A = LU$ by using elimination to find the matrix U. That is

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies E_{21}A = \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 8 & -1 & 5 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \implies E_{31}E_{21}A = \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 3 & -3 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \implies E_{32}E_{31}E_{21}A = \begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L = (E_{32}E_{31}E_{21})^{-1} = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}$$

$A\vec{x} = \vec{b} \implies LU\vec{x} = \vec{b}$. Let $U\vec{x} = \vec{y}$, where $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then,

$$L\vec{y} = \vec{b} \implies \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \implies y_1 = 1,\ y_2 = 3,\ y_3 = 3$$

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Since $U\vec{x} = \vec{y}$

$$\begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \implies x_1 = -1,\ x_2 = 3,\ x_3 = 3$$