Linear Algebra - Problem Set 1 - Solutions

Exercise I ($5 \times 5 = 25$ points)

1. The system of equations below does not have any solutions because the two lines are parallel.

$$3x_1 + 9x_2 = 15$$
$$x_2 = -3x_1 + 5$$

   False. The two lines actually intersect, so the system has a unique solution.

2. Let a be any real number. If $\vec{u} = (a, a, a)$ is perpendicular to $\vec{v}$ and $\vec{w}$, then $\vec{v}$ is parallel to $\vec{w}$.

   False. Counter-example $\vec{v} = (-2, 2, 0)$ and $\vec{w} = (3, 0, -3)$.

3. $\vec{s}$ is a linear combination of $\vec{v}$ and $\vec{w}$. of If $\vec{u}$ is perpendicular to $\vec{v}$ and $\vec{w}$, then $\vec{u}$ is perpendicular to $\vec{s}$.

   True. If $\vec{u}$ is perpendicular to $\vec{v}$ and $\vec{w}$, then $\vec{u} \cdot \vec{v} = 0$ and $\vec{u} \cdot \vec{w} = 0$. Since $\vec{s}$ is a linear combination of $\vec{v}$ and $\vec{w}$, we have $\vec{s} = \alpha\vec{v} + \beta\vec{w}$ for some $\alpha, \beta \in \mathbb{R}$. Then $\vec{u} \cdot (\alpha\vec{v} + \beta\vec{w}) = \alpha\vec{u} \cdot \vec{v} + \beta\vec{u} \cdot \vec{w} = 0$

4. If $\vec{u}$ and $\vec{v}$ are perpendicular unit vectors, then $||\vec{u} + 3\vec{v}|| = \sqrt{10}$.

   True. $\|\vec{u} + 3\vec{v}\|^2 = (\vec{u} + 3\vec{v}) \cdot (\vec{u} + 3\vec{v}) = \|\vec{u}\|^2 + 2(\vec{u} \cdot \vec{v}) + 9\|\vec{v}\|^2 = 1 + 0 + 9 = 10$.

5. The length of a unit vector in n dimensions is n.

   False. Regardless of the dimension, the length of a unit vector is always 1, by definition.

Exercise II ($3 \times 10 = 30$ points)

1. Which pairs are orthogonal among the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$? Show all your work.

   $\vec{v}_1 \cdot \vec{v}_2 = -4$
   $\vec{v}_1 \cdot \vec{v}_3 = 0 \implies \vec{v}_1 \perp \vec{v}_3$
   $\vec{v}_1 \cdot \vec{v}_4 = 2$
   $\vec{v}_2 \cdot \vec{v}_3 = 0 \implies \vec{v}_2 \perp \vec{v}_3$
   $\vec{v}_2 \cdot \vec{v}_4 = 8$
   $\vec{v}_3 \cdot \vec{v}_4 = -2$

2. Find the angle between the pairs of vectors that are not orthogonal.

$$\cos\theta_{12} = \frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\|\|\vec{v}_2\|} = \frac{-4}{\sqrt{10}\sqrt{32}} \implies \theta_{12} = 103°$$
$$\cos\theta_{14} = \frac{\vec{v}_1 \cdot \vec{v}_4}{\|\vec{v}_1\|\|\vec{v}_4\|} = \frac{2}{\sqrt{10}\sqrt{4}} \implies \theta_{14} = 72°$$
$$\cos\theta_{24} = \frac{\vec{v}_2 \cdot \vec{v}_4}{\|\vec{v}_2\|\|\vec{v}_4\|} = \frac{8}{\sqrt{32}\sqrt{4}} \implies \theta_{24} = 45°$$
$$\cos\theta_{34} = \frac{\vec{v}_3 \cdot \vec{v}_4}{\|\vec{v}_3\|\|\vec{v}_4\|} = \frac{-2}{\sqrt{4}\sqrt{4}} \implies \theta_{34} = 120°$$

3. Construct a set of 3 unit vectors out of $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$.

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix} \qquad \vec{u}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{32}}\begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix} \qquad \vec{u}_3 = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

Exercise III ($3 \times 5 = 15$ points)

Write each of the following vector equations as a matrix equation. That is, write it in the form $A\vec{x} = \vec{b}$. Specify what the matrix A, the vector $\vec{x}$, and the vector $\vec{b}$ are.

1. $x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$ $\qquad \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$

2. $x_1 \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 11 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$ $\qquad \begin{bmatrix} 2 & 5 & 1 \\ 4 & 11 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

3. $x_1 \begin{bmatrix} 3 \\ -2 \\ 8 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}$ $\qquad \begin{bmatrix} 3 & 5 \\ -2 & 0 \\ 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 8 \end{bmatrix}$

Exercise IV (20 points)

Consider the matrix below. Are the column vectors of this matrix linearly independent or linearly dependent? Show all explanations or computations.

$$\begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}.$$

To prove that a set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent, we must show that $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = 0$ has $c_1 = c_2 = c_3 = 0$ as its only solution. Thus, we must solve:

$$c_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 5 \\ 0 & \boxed{1} & 4 \\ 0 & 2 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The last row indicates that $0 = 0$, which means that we can choose c_3 to be any nonzero constant. The second row gives $c_2 = -4c_3$, and the first row gives $c_1 = -5c_3$. Thus, we can find nonzero c_1, c_2, c_3 that solve the above system. Hence, the column vectors of the matrix are linearly dependent.

Exercise V (10 points)

Suppose that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly dependent set of vectors in $\mathbb{R}^n$ and $\vec{v}_4$ is vector in $\mathbb{R}^n$. Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is also a linearly dependent set.

Applying the definition of linearly dependent to $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ implies that there exist scalars c_1, c_2, and c_3, not all zero, such that $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Adding $0\vec{v}_4 = \vec{0}$ to both sides of this equation results in

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + 0\vec{v}_4 = \vec{0}.$$

c_1, c_2, c_3 and 0 are not all zero, the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ satisfies the definition of a linearly dependent set.