Exam 1 practice problems

Question 1: What is the difference between $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$, $\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$, and $\text{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\}$?

Question 2: Construct a matrix $\underset{\sim}{A}$ such that for all $\vec{v} \in \mathbb{R}^3$, $\underset{\sim}{A}\vec{v} \in \text{span}\left\{\vec{0}\right\}$.

Question 3: Prove that any elimination E_{21} matrix size $n \times n$ is invertible.

Question 4: Prove a matrix with a single 1 in each column and row (and zeros everywhere else) has an inverse.

Question 5: If $\vec{v}_2 \notin \text{span}\{\vec{v}_1\}$ is $\{\vec{v}_1, \vec{v}_2\}$ a linearly independent set? Why or why not? ($\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$)

Related question: If $v_1 \in \text{span}\{v_2\}$ is $v_2 \in \text{span}\{v_1\}$? Why or why not?

Question 6: If $\vec{v}_2 \notin \text{span}\{\vec{v}_1\}$ and $\vec{v}_1 \notin \text{span}\{\vec{v}_2\}$ is $\{\vec{v}_1, \vec{v}_2\}$ a linearly independent set? Why or why not? ($\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$)

Question 7: State whether the dot product is positive, negative, or zero. What are the possible outcomes? ($\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^n$)

7a Suppose $v_1 \perp v_2$, $||v_1|| > 0$ and $||v_2|| > 0$.

What is the sign of $(v_1 + v_2) \cdot v_1$?

7b Suppose $v_1 \perp v_2$, $||v_1|| > 0$ and $||v_2|| > 0$.

What is the sign of $(v_1 - v_2) \cdot v_2$?

7c Suppose $(v_1 + v_2) \cdot v_3 = 0$ and $||v_1|| > 0$, $||v_2|| > 0$, and $||v_3|| > 0$. If $v_1 \perp v_3$ is $v_3 \perp v_2$?

Question 8: When is the following matrix singular? $\begin{bmatrix}1 & a\\ b & 1\end{bmatrix}$

Question 9: Is the matrix product $(AA^T)(A^TA)$ symmetric? Why or why not? $(A \in \mathbb{M}_{n\times m})$

Question 10: For what matrices it is possible to perform LU factorization? Which matrices require a permutation in order to complete the factorization?

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Question 10(b): Which of the following products have an LU factorization?

$$AB, BA, AC, CA, DB, CD$$

Solutions:

Question 1: What is the difference between $\text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$, $\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$, and $\text{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\}$?

Line in $\mathbb{R}^2$, line in $\mathbb{R}^3$, line in $\mathbb{R}^4$

Question 2: Construct a matrix $\underline{A}$ such that for all $\vec{v} \in \mathbb{R}^3$, $\underline{A}\vec{v} \in \text{span}\left\{\vec{0}\right\}$.

$A = \underline{0}_{3\times 3}$

Question 3: Prove that any elimination E_{21} matrix size $n \times n$ is invertible.

Use row reduction to reduce E_{21} to I ($[A|I] \to [I|A^{-1}]$). Or set B to be the inverse, show $BE_{21} = I$.

Question 4: Prove a matrix with a single 1 in each column and row (and zeros everywhere else) has an inverse.

Use row reduction to reduce P to I ($[P|I] \to [I|P^{-1}]$). Or set B to be the inverse, show $BP = I$.

Question 5: If $\vec{v}_2 \notin \text{span}\{\vec{v}_1\}$ is $\{\vec{v}_1, \vec{v}_2\}$ a linearly independent set? Why or why not? $(\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n)$

Related question: If $v_1 \in \text{span}\{v_2\}$ is $v_2 \in \text{span}\{v_1\}$? Why or why not?

No, let $\vec{v}_1 = (0, 0)$ and $\vec{v}_2 = (1, 1)$. $\vec{v}_2 \notin \text{span}\{\vec{v}_1\}$ but these vectors are a linearly dependent set (show they are linearly dependent set).

Question 6: If $\vec{v}_2 \notin \text{span}\{\vec{v}_1\}$ and $\vec{v}_1 \notin \text{span}\{\vec{v}_2\}$ is $\{\vec{v}_1, \vec{v}_2\}$ a linearly independent set? Why or why not? $(\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n)$

Yes, suppose for the sake of contradiction, that $\{\vec{v}_1, \vec{v}_2\}$ is a linearly dependent set, then there exists a $c_1 \neq 0$ or $c_2 \neq 0$ such that

$$c_1 v_1 + c_2 v_2 = 0$$

.

Suppose $c_1 \neq 0$, then $v_1 = \frac{-c_2}{c_1} v_2$, therefore $\vec{v}_1 \in \text{span}\{\vec{v}_2\} \Rightarrow\Leftarrow$.

Suppose $c_2 \neq 0$, then $v_2 = \frac{-c_1}{c_2} v_1$, therefore $\vec{v}_2 \in \text{span}\{\vec{v}_1\} \Rightarrow\Leftarrow$.

Therefore, $c_1 = 0$ and $c_2 = 0$, therefore $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set.

Question 7: State whether the dot product is positive, negative, or zero. What are the possible outcomes? ($\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^n$)

7a Suppose $v_1 \perp v_2$, $||v_1|| > 0$ and $||v_2|| > 0$.

What is the sign of $(v_1 + v_2) \cdot v_1$?

$(v_1 + v_2) \cdot v_1 = v_1 \cdot v_1 + v_2 \cdot v_1 = v_1 \cdot v_1 > 0$

7b Suppose $v_1 \perp v_2$, $||v_1|| > 0$ and $||v_2|| > 0$.

What is the sign of $(v_1 - v_2) \cdot v_2$?

$(v_1 - v_2) \cdot v_2 = v_1 \cdot v_2 - v_2 \cdot v_2 = -v_2 \cdot v_2 < 0$

7c Suppose $(v_1 + v_2) \cdot v_3 = 0$ and $||v_1|| > 0$, $||v_2|| > 0$, and $||v_3|| > 0$. If $v_1 \perp v_3$ is $v_3 \perp v_2$?

$(v_1 + v_2) \cdot v_3 = v_1 \cdot v_3 + v_2 \cdot v_3 = v_2 \cdot v_2 = 0$ therefore $v_3 \perp v_2$

Question 8: When is the following matrix singular? $\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$

When $ab = 1$

Question 9: Is the matrix product $(AA^T)(A^TA)$ symmetric? Why or why not? ($A \in \mathbb{M}_{n\times m}$)

$$[(AA^T)(A^TA)]^T = (A^TA)^T(AA^T)^T = A^TAAA^T$$

Cannot compute unless $n = m$. If $n = m$ then you still can't show its symmetric, the product of two symmetric matrices is not necessarily symmetric.

example let $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

Question 10: For what matrices it is possible to perform LU factorization? Which matrices require a permutation in order to complete the factorization?

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

A has no LU factorization. $PA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has an LU factorization.

$L = I, \quad U = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$

B has a LU factorization. $L = I$, $U = B$.

C has a LU factorization. $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad U = I.$

D has a LU factorization. $L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

LU decomposition technicalities

Question 10(b): Which of the following products have an LU factorization?

$$AB, BA, AC, CA, DB, CD$$

AB yes, BA yes, AC yes, CA no, DB yes, CD yes.

Test is entry a_{11} zero? If a_{11} is zero, is a_{21} zero?